LARGE DEFLECTION ANALYSIS OF SQUARE ISOTROPIC PLATES STIFFENED AROUND THE BOUNDARIES

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ABSTRACT

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The objective of the present work is to investigate the behavior of thin plates stiffened around the boundaries and subjected to uniformly distributed normal load. The stiffeners are assumed to be thin with negligible torsional stiffness and the plane of the plate.

The non-linearity effects caused by the large deflection of the plate are studied and results are obtained for the deflections, the in-plane displacements and the stress distribution across the plate. Representative non-dimensional solutions are given to show the effect of membrane action and that of the edge shear restraints.

The static nonlinear system has been idealized by a system of nonlinear cubic algebraic equations by means of the variational techniques. It is believed that this is the first attempt to employ the variational techniques in solving plate problems with moving boundaries.

A comparison between the results of this work and a few existing large deflection solutions for non-stiffened plates are made and very good agreement is found for the displacements and the deflections.

ACKNOWLEDGEMENTS

I wish to place on record my sincere thanks and gratitude to Professor C. Marsh and Professor T. S. Sankar for their guidance and suggestions throughout this project.

I would also like to take this opportunity of expressing my thanks to Dr. K. Ha for his encouragement.

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NOMENCLATURE

A_s	Cross-sectional area of stiffener
a	Length of plate
D	Flexural rigidity of the plate
E	Young's modulus of elasticity
K	Extensional rigidity of the plate due to development of membrane stresses defined by Equation (5.27)
M_x, M_y	Bending moments in the x and y directions
M_{xy}	Twisting moment about the x and y axis
N_x, N_y, N_{xy}	Membrane forces per unit width
q	Lateral load applied to the plate
Q	Non-dimensionalized load
S	Plate area = at
t	Thickness of plate
u,v,w	Displacements of the points in the middle plane of plate in the x,y and z-directions, respectively
U	Strain energy of the system
U_m	Strain energy due to membrane action
U_b	Strain energy due to bending
U_s	Strain energy due to compression in stiffeners
V	Potential energy of the load
x,y,z	Cartesian coordinates with origin at centre of plate
γ	Shear strain

δ	Deflection at the centre of the plate
D_N	Non-dimensionalized deflection at the centre of the plate
ξ, η	Non-dimensionalized coordinate in the x and y-directions
ε	Direct strain
π	Total potential energy of the system
ϕ	Stress function (Airy's stress function)
ν	Poisson's ratio
σ_{xb}, σ_{yb}	Extreme-fiber bending stresses in the x and y-directions respectively
σ_{xm}, σ_{ym}	Median-fiber membrane stresses in the x and y-directions respectively
τ_{xyb}	Extreme-fiber shear stress
τ_{xym}	Membrane shear stress
$\Delta\Delta(\ )$	$\partial^4(\ )/\partial x^4 + 2[\partial^4(\ )/\partial x^2\partial y^2] + \partial^4(\ )/\partial y^4$
$\nabla^2(\ )$	$\partial^2(\ )/\partial x^2 + \partial^2(\ )/\partial y^2$
$\delta(\ )$	A virtual displacement in the indicated variable

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* Tensile loads, stresses and strains are positive.

CHAPTER I

INTRODUCTION

CHAPTER 1

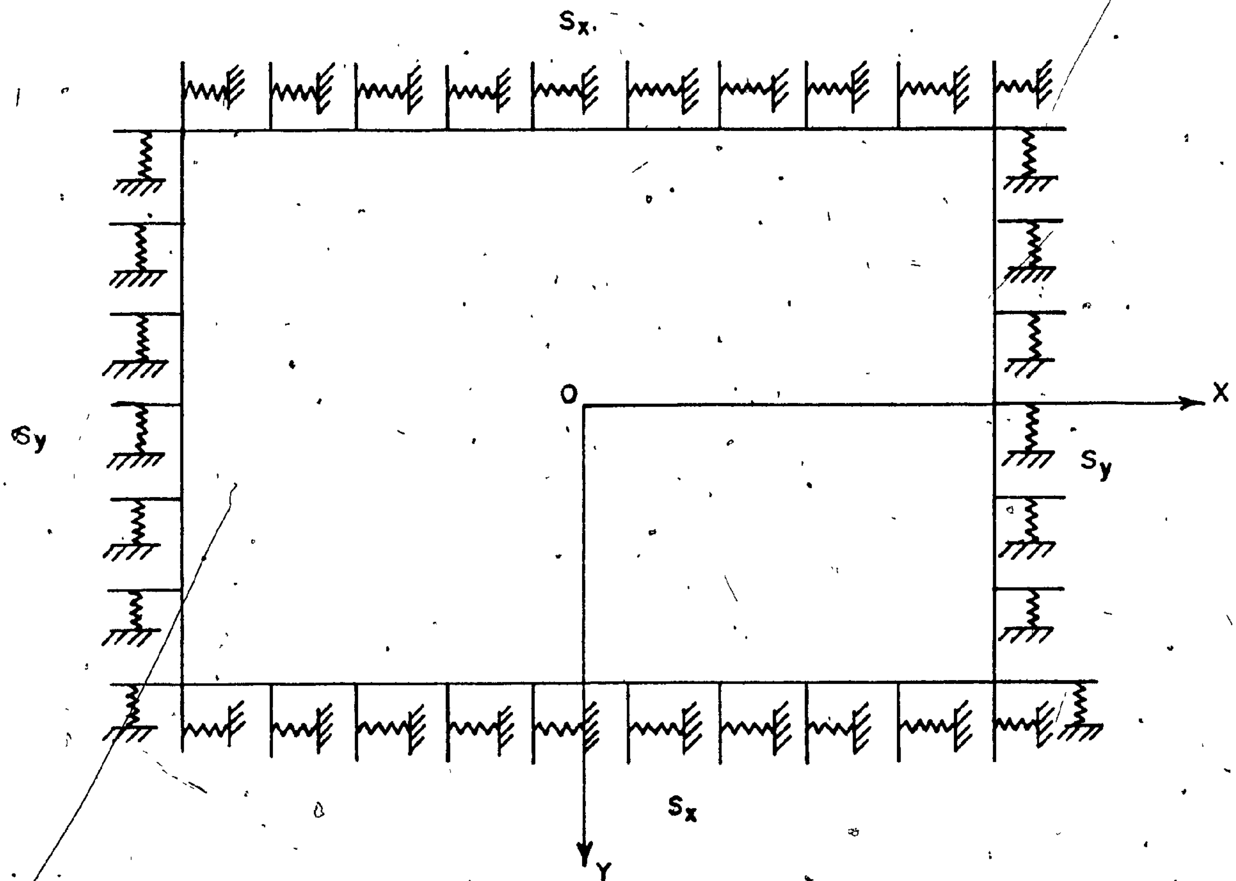
INTRODUCTION

Unstiffened plates subjected to lateral loading undergo, in general, considerable deflection before the onset of plasticity and, as a result, develop membrane stresses while still well within the elastic range. In practice, plates are seldom used in isolation as they are usually connected at the boundaries to adjoining structures. The effect of this constraint can be approximately represented by boundary stiffeners giving elastic restraint to in-plane shear only, as modelled in Fig. 1.1.

The analysis with edge shear restraints permits not only a better assessment of the boundary effects than has so far been possible, but also the determination of the magnitudes of the restraint which for practical purposes may be considered to be effective.

Edge restraints are provided in practice by the equivalent of stiffeners around the boundaries, as shown in Fig.1.2.

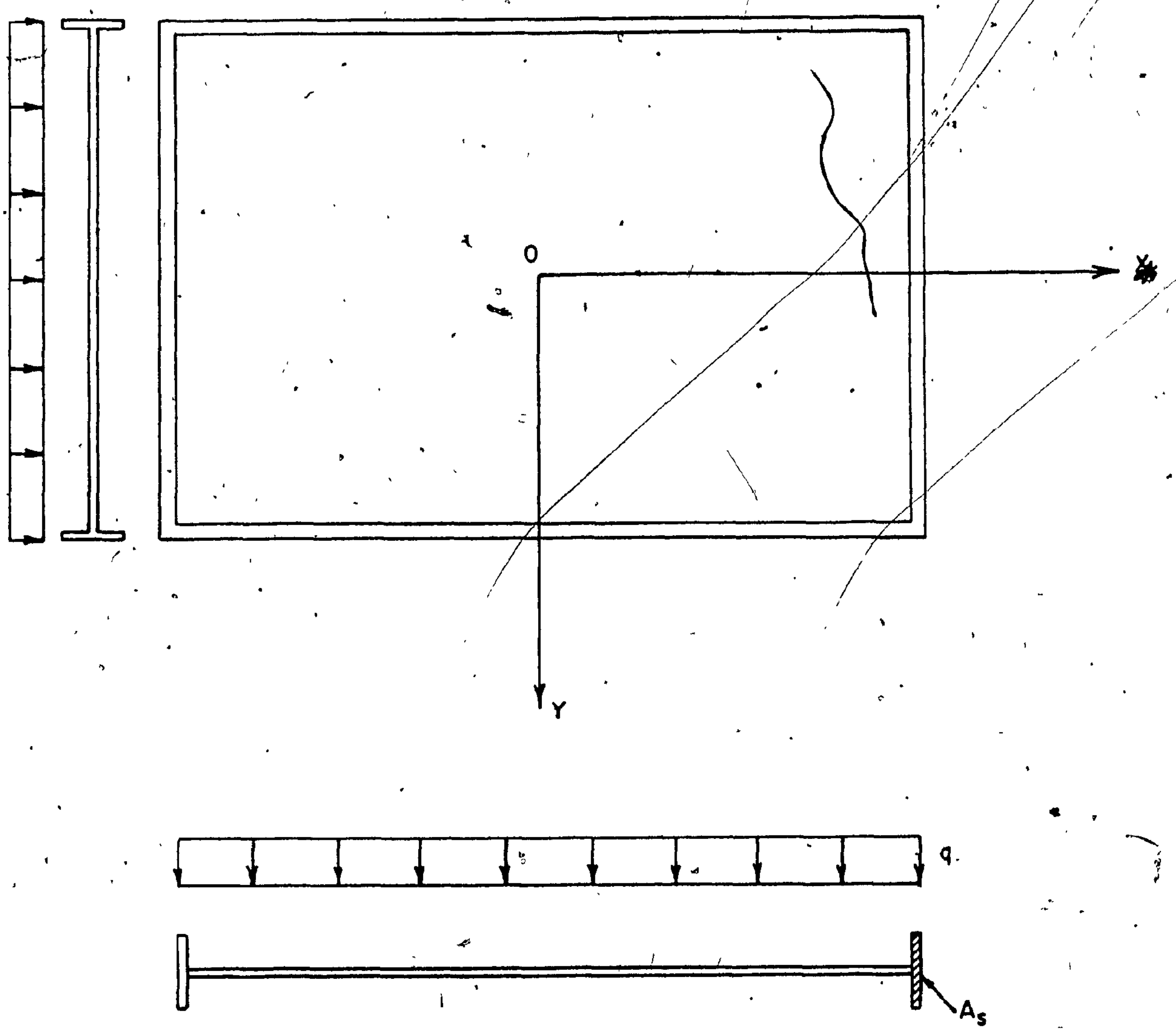
The cross-sectional area of the edge stiffeners is varied and may take any value between zero and infinity corresponding respectively to the condition of complete freedom from restraint and to the condition of complete fixity at the edges against movement parallel to the edge (stiffeners with very large area.)



SHEAR ELASTIC RESTRAINTS

(restraint against movement parallel to edge)

FIG (1.1)



RECTANGULAR PLATE WITH BOUNDARY STIFFENERS

FIG (1.2)

The square plate, which is of thickness t, occupies the region $-\frac{a}{2} \leq x \leq \frac{a}{2}$ and $-\frac{a}{2} \leq y \leq \frac{a}{2}$ and $|z| \leq \frac{t}{2}$. The displacements of the midplane $z = 0$ in the x,y and z directions are denoted by $u(x,y)$, $v(x,y)$ and $w(x,y)$. The midplane stresses are $\sigma_{xm}(x,y)$, $\sigma_{ym}(x,y)$ and $\tau_{xym}(x,y)$ and the bending moments are $M_x(x,y)$, $M_y(x,y)$ and $M_{xy}(x,y)$.

The present work gives a solution of the non-linear, large deflection problem of the deformation of uniformly loaded thin square plates stiffened along their edges by elastically compressible stiffeners which are flexible in the plane of the plate.

The influence of the variation of the cross-sectional area of the boundary stiffeners on the stress distribution is also examined.

1.1 ASSUMPTIONS

The assumptions made in the present analysis are as follows:

1) The boundaries are simply supported to hold, and the stiffeners possess negligible torsional stiffness and negligible flexural stiffness in the plane of the plate, but appreciable shear and compression stiffness.

2) When loaded, the edges of the plate will have both in-plane and angular displacements relative to each other, while the stiffeners will follow the plate edges as the normal membrane stress at the edges is zero.

3) The plate is simply supported and has no initial curvature.

4) The plate and the stiffeners are made of an isotropic linearly elastic material.

5) The plate is thin relative to other dimensions.

6) The sources of strain energy considered are bending and torsion of the plate, membrane action in the plate and axial stresses in the stiffeners.

7) Deflections remain small relative to the dimensions (a,b) of the plate.

8) In the derivation of the boundary conditions, it is assumed that the elastic edge restraints and the loading are symmetrical about both center lines of the plate.

CHAPTER II

A REVIEW OF THE LITERATURE

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A REVIEW OF THE LITERATURE

The classical small deflection theory of plates, widely used because of its simplicity, takes no account of the stretching of the middle plane of the plate, which with the exception of developable surfaces, occurs as a necessary consequence of the lateral displacement. The fundamental equations describing this large deflection behaviour were derived by Von Karman [3] in 1910. Consideration of the membrane stresses introduces non-linear terms into the differential equations, which increase the complexity of the problem significantly. In consequence, comparatively few investigators have produced satisfactory solutions to the large deflection problem.

To the knowledge of the author, the first solution of the classical large deflection equations, which permits the plate edges to distort is that by Kaiser [5] in 1936. Kaiser converted Von Karman's differential equations into difference equations, and calculated deflections and stresses for a square plate under constant pressure, assuming simply supported edges with zero membrane stresses, using the step-by-step method. In the step-by-step method, as used by Kaiser, the problem of uncertainty about convergence especially at higher loads was encountered. An explanation of the failure to converge in the step-by-step method may be thought of as a solution to the linear bending problem of the plate subjected to a non-uniform load. This load consists of the uniform normal pressure q modified by the non-uniform normal force produced by the interaction of the membrane stresses with the deflection arising from the previous step. The stresses and deflections obtained as a result become significantly larger than previous iterations, and these, in turn, produce even larger disturbances. This process is self-propagating, swamping the original solution and resulting in divergence.

Levy [10], in 1941, presented an analytical solution for Von Karman's fundamental equations for large deflections of plates, applied for the case of simply supported square plate under lateral loading.

Levy expressed the deflection and the stress function in terms of two infinite Fourier series, the coefficients of which were subsequently obtained by substituting the series into the governing equations and solving a set of simultaneous cubic algebraic equations. The assumed series for the stress function makes the method inapplicable to plates having zero normal stresses or non-zero shear stresses at the edges. Therefore the case of Levy's solution with the rather unusual boundary conditions, (occasioned by his choice of Fourier series), made it somewhat restricted in its applications, and is limited to the case of zero shear stress and constant normal displacement along the boundaries, with no net force applied in the plane of the plate.

Bauer, Bauer, Becker and Reiss [12], in 1965, applied an iterative procedure to obtain a numerical solution for the Von Karman equations for rectangular plates, subjected to uniform normal pressure. On the simply supported boundaries, it was assumed that the normal membrane stresses and the tangential membrane displacement vanish.

In their paper they reduced Von Karman equations to a coupled system of four second-order partial differential equations. The four dependent variables are $w(x,y)$, the stress function and their Laplacians. Their numerical solutions were obtained by an iteration method which employs an acceleration parameter θ. If $\theta = 1$ the iterations are simple iterations, and they converge only if q is quite small. For larger values of q, it is essential to employ $\theta < 1$ to achieve convergence.

Hartman, Kao and Guzman-Barron [11], discussed the effect of uniform edge force on the large deflection of transversely loaded plates.

Von Karman equations were solved iteratively using finite difference methods. The finite difference equations are solved in such a manner that $\nabla^4 W$ and $\nabla^4 \phi$ can be analyzed as linear algebraic equations in each step of iteration. In their analysis, the edge forces N_1 and N_2 are assumed to be uniform and the transverse load q is increased in steps until the maximum deflection in the center reaches about 2.5 times the plate thickness. The maximum deflection obtained at the final step, was for a non-dimensionalized load $Q = \frac{q}{E}(\frac{a}{t})^4$ of 120, the deflection of 2.5t is low relative to other investigators."

Djubek [1] in 1966, presented an analytical solution for the Von-Karman's equations. The equations were solved by the method of P.E. Papkovich, implying an exact satisfaction of the condition of compatibility and solving the first equation approximately by the Galerkin method. He expanded the deflection function in terms of the Fourier series, and the stress function is considered as the sum of the biharmonic function and the particular integral of the second equation of Von Karman's system. Numerical results showing the effect of flexural and normal rigidity of stiffeners are given for a square plate loaded by shear and compression.

Basu and Chapman [17], in 1966, gave a treatment for the large deflection behaviour of rectangular orthotropic plates under uniformly distributed load, and having elastic flexural and extensional edge restraints. Basu and Chapman formulated the problem in terms of the deflection w and the Airy stress function. The problem can be defined as two biharmonic equations, the first in terms of the deflections with the right-hand side involving the load plus differentials of the stress function and the deflection, while the second equation is the biharmonic of the stress function with the right-hand side involving differentials of the deflections.

The finite difference forms of these equations are solved using the modified step-by-step method in which the values of the function obtained from the solution of one equation are substituted into the right-hand side of the other equation. The step-by-step procedure is repeated until the change in the function between the successive solutions is less than by 1%. To obtain convergence in the iterative process, Basu and Chapman used the value of $(W)^{j,s+1}_{mean}$ instead of $(W)^{j,s+1}$, where $(W)^{j,s+1}_{mean}$, (the non-dimensional total deflection at p nodes obtained for a load $(Q)^{j}$ after s cycles), is equal to the average of $(W)^{j,s}$ and $(W)^{j,s+1}$.

The step-by-step technique suffers from two main disadvantages, the first is that each step requires a solution of the two sets of finite difference equations involving the inversion of matrices. The second is the more serious disadvantage, as there is no certainty that the correct solution has been obtained. The criterion used to determine the number of steps is that there must be a change of less than 1% between the successive steps, but this does not guarantee that the values within 1% of the correct finite difference solution have been obtained. This is because the convergence is from below and the equations are ill-conditioned. It is quite possible that the changes between the successive steps may be small but that the values are still several percentages from the true solution to the finite difference equations.

The effect of varying the Poisson's ratio from .316 to .3 on the behaviour of a uniformly loaded square isotropic plate was computed by Basu and Chapman and found to be negligible.

A general method of studying the deflection of variable thickness elastic plates using the dynamic relaxation method was described by K.R.Ruston [6,13] in 1968. The non-linear terms arising in the equations due to the large deflection of plates can be included directly by the iterative finite difference technique on which the dynamic relaxation method is based.

The Von Karman equations for the large deflection of plates are solved by the dynamic relaxation method. Detailed results are presented for square plates having simply supported edges with zero in-plane boundary stresses.

K.R.Ruston obtained the dynamic relaxation equations by adding to the Von Karman equations the acceleration term and the viscous damping term. The concept of introducing time with acceleration and viscous damping is a convenient way of giving the technique a physical meaning. In the dynamic relaxation solution the values oscillate about the static value and their convergence to the static value can be traced. This way, the convergence of the dynamic relaxation method is more certain than the step-by-step method of Basu and Chapman, however it takes a longer computation time.

Brown and Harvey [2] made a theoretical analysis for the problem of flat rectangular plates undergoing large deflections due to the action of a uniform lateral pressure. The governing partial differential equations were replaced by their finite difference equivalents and the resulting difference equations were solved by the over-relaxation method. The tolerance used in the iterative process was .01 of the plate thickness.

To force the solutions to converge, Brown and Harvey used an interpolated iteration to bias the assumed value for each iteration towards the previous iterate, thus reducing the non-uniform normal force at each step and preventing the solution from diverging.

In 1969, Scholes and Bernstein [7] discussed another way of formulation of finite difference equations using an approximation to the potential energy rather than to the differential equations. This method, which does not seem to have been previously applied, is akin to the finite element technique and offers several advantages over more conventional finite difference approach.

The Schole and Bernstein method consists of replacing the potential energy by an approximation. A grid work is defined over the plate, dividing it into squares. The integral of the energy for each square is replaced by its formulation as a finite difference quantity averaged over a square or alternatively the integration is performed inside the square for an assumed displacement distribution which is defined by the displacements at the corners of the square. The integration over all the squares is now replaced by a summation of each integral multiplied by the area of the square over which it is averaged. Then the resulting expression is minimized by equating to zero its partial derivatives with respect to each of the finite difference variables which are the displacements at each node. The linear matrices resulting from this procedure are always positive,definite and symmetric and are thus open to solution by a wide range of numerical methods.

Föpple [4] represented a solution for a simply supported rectangular plate with the boundary conditions that the in-plane displacements u and v vanish at the plate boundaries.

His elegant method consists of a combination of the known solutions given by the theory of small deflections and the membrane theory. He assumed that the load q can be resolved into two parts q_1 and q_2 in such a manner that part q_1 is balanced by the bending and shearing stresses calculated by the theory of small deflections, part q_2 being balanced by the pure membrane stresses. Unfortunately, this method cannot be used, as a pure membrane with zero normal edge stresses is an unstable structure.

CHAPTER III

LARGE DEFLECTION ANALYSIS

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LARGE DEFLECTION ANALYSIS

3.1 MEMBRANE STRAINS (IN-PLANE STRAINS)

When a plate deflects into a non-developable surface the middle plane of the plate will no longer be a neutral plane and membrane strains will be developed because of the difference in length between the straight boundary and the curved parts of the surface. Considering a linear element AB of the middle plane in the x-direction, it may be seen from Fig. 3.1 that the elongation of the element due to the displacement u is equal to $(\frac{\partial u}{\partial x})dx$. The elongation of the element due to the displacement w is

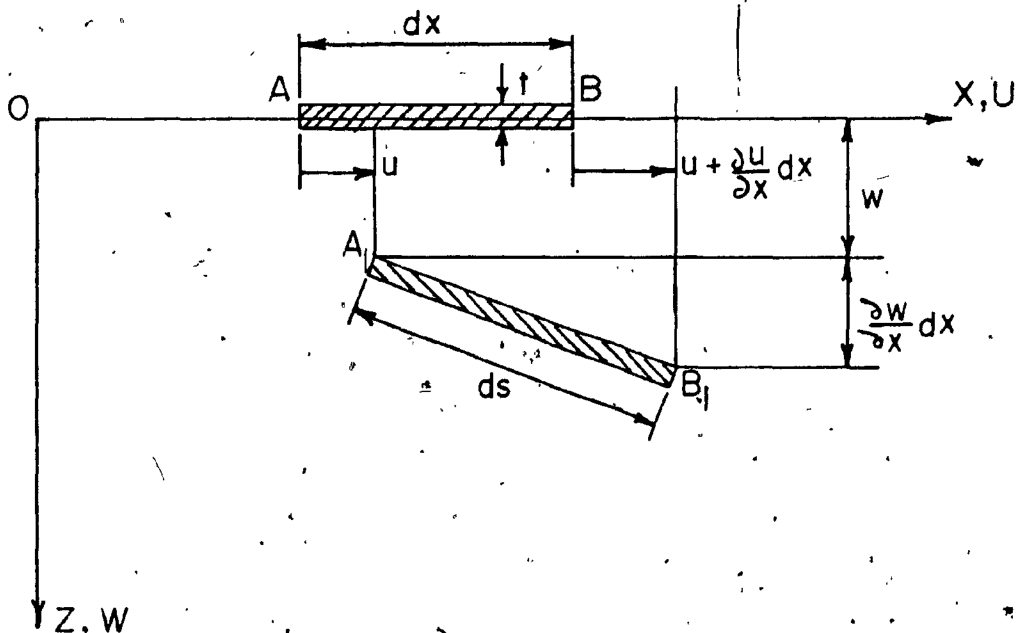
$$ds - dx = [\sqrt{1 + (\frac{\partial \omega}{\partial x})^2} - 1]dx \approx \tfrac{1}{2}(\frac{\partial \omega}{\partial x})^2 \quad dx$$

Thus the total unit elongation in the x-direction of an element taken in the middle plane of the plate is

$$\varepsilon_x = \varepsilon_x^L + \varepsilon_x^{NL} = \frac{\partial u}{\partial x} + \tfrac{1}{2}(\frac{\partial \omega}{\partial x})^2 \qquad (3.1)$$

Similarly, the strain in the y-direction is

$$\varepsilon_y = \varepsilon_y^L + \varepsilon_y^{NL} = \frac{\partial v}{\partial y} + \tfrac{1}{2}(\frac{\partial \omega}{\partial y})^2 \qquad (3.2)$$



$$\epsilon^{L} = \frac{\partial u}{\partial x}$$

$$\epsilon_x^{NL} = \frac{ds - dx}{dx} = \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \approx \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

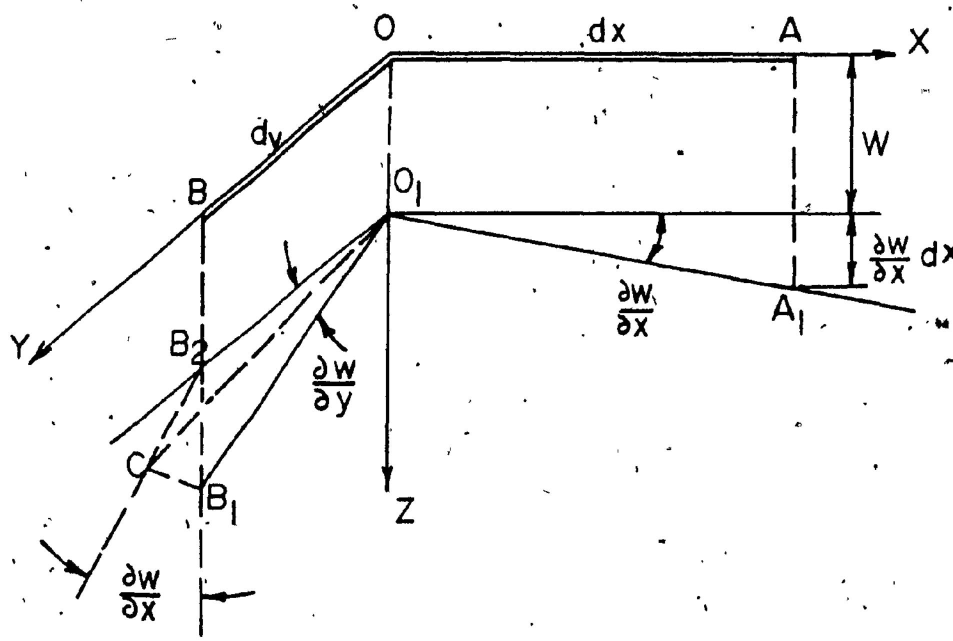
STRAIN, ϵ_x DUE TO DEFLECTIONS

FIG (3.1)

Considering now the shear strain in the middle plane due to bending, we conclude as before that the shear strain due to the displacements u and v is $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. To determine the shearing strain due to the displacement ω we take two infinitely small linear elements OA and OB in the x and y directions, as shown in Fig. 3.2. Because of the displacements in the z-direction, these elements take the positions O_1A_1 and O_1B_1. The difference between the angle $\frac{\pi}{2}$ and the angle $A_1O_1B_1$ is the shear strain corresponding to the displacement ω.

To determine this difference we consider the right-angle $B_2O_1A_1$, in which B_2O_1 is parallel to BO. Rotating the plane $B_2O_1A_1$ about the axis O_1A_1 by the angle $\frac{\partial \omega}{\partial y}$, we bring the plane $B_2O_1A_1$ into coincidence with the plane $B_1O_1A_1$ and the point B_2 to position C. The displacement BC is equal to $\frac{\partial \omega}{\partial y}\,dy$ and is inclined to the vertical B_2B_1 by the angle $\frac{\partial \omega}{\partial x}$. Hence B_1C is equal to $\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}\,dy$, and the angle CO_1B_1, which represents the shear strain corresponding to the displacement ω is $\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}$. Adding this shear strain to the strain produced by the displacements u and v we obtain

$$\gamma_{xy} = \gamma^{L}_{xy} + \gamma^{NL}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial x}\,\frac{\partial \omega}{\partial y} \qquad (3.3)$$



STRAIN, γ_{xy} DUE TO DEFLECTION

FIG(3.2)

3.2 BENDING AND MEMBRANE STRESSES

Consider the equilibrium of a small element cut from the plate by two pairs of planes parallel to the xz and yz coordinate planes, Fig. 3.4. In addition to the forces shown in Fig. 3.3, we now have forces acting in the middle plane of the plate. We denote the magnitude of these forces per unit length by N_x, N_y and $N_{xy} = N_{yx}$, as shown in the Figure:

Projecting these forces on the x and y axes, and assuming that there are no body forces or tangential forces acting in those directions at the faces of the plate, we obtain the following equations of equilibrium.

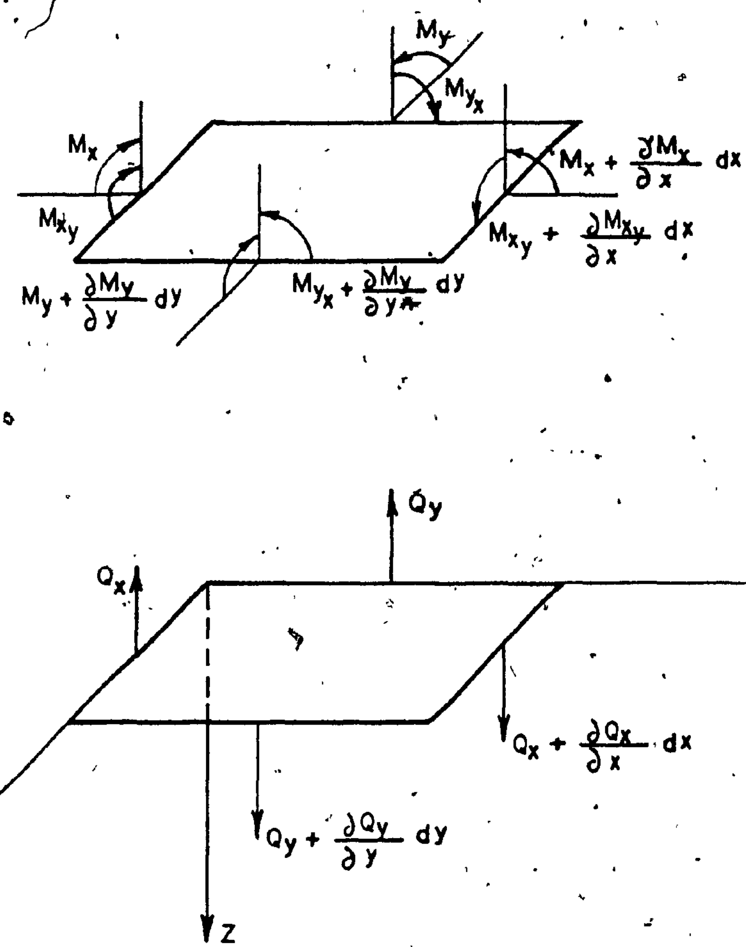
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{3.4}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{3.5}$$

These equations are independent of the three equations of equilibrium considered for the case of small deflection-bending theory given below:

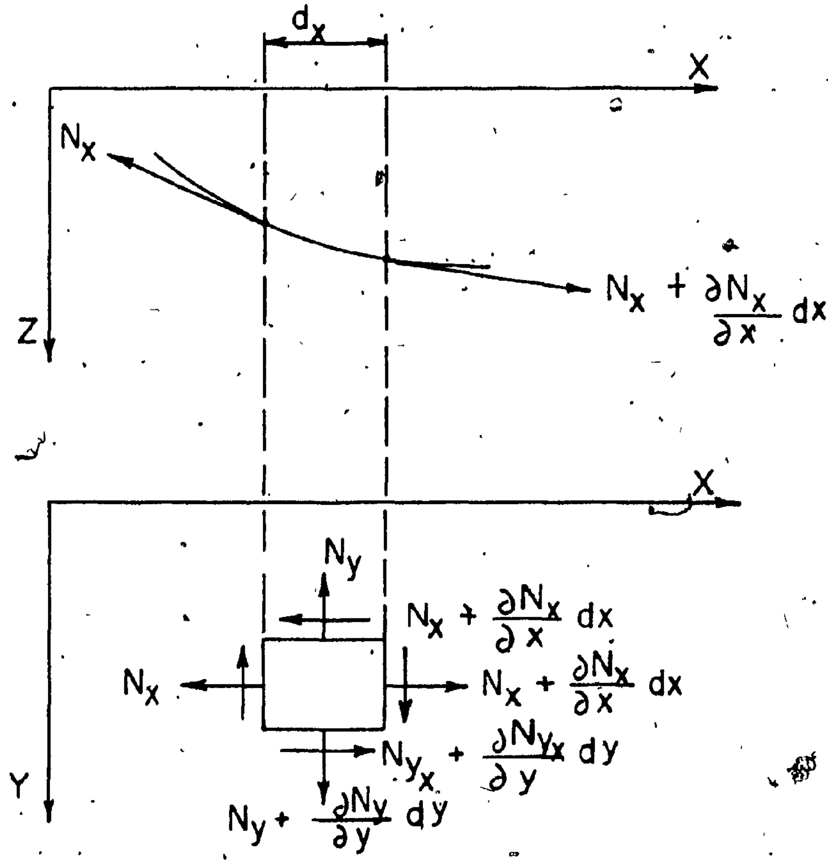
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{3.6}$$

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \tag{3.7}$$



BENDING STRESSES

FIG (3.3)



MEMBRANE STRESSES

FIG (3.4)

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \qquad (3.8)$$

In considering the projection of the forces shown in Fig.3.4, on the z-axis, we must take into account the bending of the plate and the resulting small angles between the forces N_x and N_y that act on the opposite sides of the element. As a result of this bending, the projection of the normal forces N_x on the z-axis gives:

$$- N_x \, dy \frac{\partial \omega}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\frac{\partial \omega}{\partial x} + \frac{\partial^2 \omega}{\partial x^2} dx\right) dy$$

After simplification, if the small quantities of higher than the second-order are neglected, this projection becomes

$$N_x \frac{\partial^2 \omega}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial \omega}{\partial x} dxdy \qquad (3.9)$$

In the same way the projection of the normal forces N_y on the z-axis gives

$$N_y \frac{\partial^2 \omega}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y} \frac{\partial \omega}{\partial y} dxdy \qquad (3.10)$$

Regarding the projection of the shear forces N_{xy} on the z-axis, we observe that the slope of the deflected surface in the y-direction on the two opposite sides of the element is $\frac{\partial \omega}{\partial y}$ and $\frac{\partial \omega}{\partial y} + \left(\frac{\partial^2 \omega}{\partial x \partial y}\right) dx$.

Hence, the projection of the shear forces on the z-axis is equal to

$$N_{xy} \frac{\partial^2 \omega}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial \omega}{\partial y} dxdy \tag{3.11}$$

An analogous expression can be obtained for the projection of the shear force $N_{yx} = N_{xy}$ on the z-axis. The final expression for the projection of all the shear forces on the z-axis, then can be written as

$$2N_{xy} \frac{\partial^2 \omega}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial \omega}{\partial y} dxdy + \frac{\partial N_{xy}}{\partial y} \frac{\partial \omega}{\partial x} dxdy \tag{3.12}$$

Eliminating the shear forces Q_x and Q_y from Equations (3.6), (3.7) and (3.8), and observing that $M_{yx} = - M_{xy}$ (by virtue of $\tau_{xy} = \tau_{yx}$), the equation of equilibrium, of the small deflection theory of plates, is in the form:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = - q \tag{3.13}$$

Adding expressions (3.9), (3.10) and (3.12) to the load q dxdy, acting on the element and using equations (3.4) and (3.5), we obtain, instead of equation (3.13), the following equation of equilibrium:

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - (q + N_x \frac{\partial^2 \omega}{\partial x^2} + $$

$$+ N_y \frac{\partial^2 \omega}{\partial y^2} + 2N_{xy} \frac{\partial^2 \omega}{\partial x \partial y}) \tag{3.14}$$

The relations between the deflected surface and the bending moment are given by:

$$M_x = - D\left[\frac{\partial^2\omega}{\partial x^2} + \nu \frac{\partial^2\omega}{\partial y^2}\right] \qquad (3.15)$$

$$M_y = - D\left[\frac{\partial^2\omega}{\partial y^2} + \nu \frac{\partial^2\omega}{\partial x^2}\right] \qquad (3.16)$$

$$M_{xy} = - M_{yx} = D(1-\nu)\left(\frac{\partial^2\omega}{\partial x\partial y}\right) \qquad (3.17)$$

where $D = Et^3/12(1-\nu^2)$

Substituting expressions (3.15), (3.16) and (3.17) into Eq.(3.14) gives:

$$\frac{\partial^4\omega}{\partial x^4} + 2\frac{\partial^2\omega}{\partial x^2\partial y^2} + \frac{\partial^4\omega}{\partial y^4} = \frac{1}{D}\left[q + N_x\frac{\partial^2\omega}{\partial x^2} + N_y\frac{\partial^2\omega}{\partial y^2} + 2N_{xy}\frac{\partial^2\omega}{\partial x\partial y}\right] \qquad (3.18)$$

3.3 GENERAL EQUATIONS FOR LARGE DEFLECTION OF PLATES

In discussing the behaviour of plates, we use Eq.(3.18) which was derived by considering the equilibrium of an element of the plate in the direction perpendicular to the plate.

The forces N_x, N_y and N_{xy} governed by equations (3.4) and (3.5), depend not only on the external forces applied in the xy-plane, but also on the strain of the middle plane of the plate due to bending.

The third equation necessary to determine the three quantities N_x, N_y, and N_{xy} is obtained from a consideration of the strain in the middle surface of the plate during bending. The corresponding strain components are given by Equations (3.1),(3.2) and (3.3).

By taking the second derivatives of these expressions and combining the resulting expression, it can be shown that

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 \omega}{\partial x^2}\right)\left(\frac{\partial^2 \omega}{\partial y^2}\right) \tag{3.19}$$

Replacing the strain components by:

$$\varepsilon_x = \frac{1}{tE}\left[N_x - \nu N_y\right] \tag{3.20}$$

$$\varepsilon_y = \frac{1}{tE}\left[N_y - \nu N_x\right] \tag{3.21}$$

$$\gamma_{xy} = \frac{1}{tG} N_{xy} \tag{3.22}$$

A Further equation in terms of N_x,N_y and N_{xy} is obtained. The solution of equations (3.4), (3.5) and this additional equation gives the quantities N_x,N_y and N_{xy}. The solution of these three equations is greatly simplified by the introduction of a stress function. The stress function $\phi = \phi(x,y)$, defined by equations (3.23),(3.24) and (3.25) below satisfies equation (3.4) and (3.5):

$$N_x = t\frac{\partial^2\phi}{\partial y^2} \tag{3.23}$$

$$N_y = t\frac{\partial^2\phi}{\partial x^2} \tag{3.24}$$

$$N_{xy} = -t\frac{\partial^2\phi}{\partial x\partial y} \tag{3.25}$$

If the above equations are substituted in equations (3.20),(3.21) and (3.22), the strain components become

$$\varepsilon_x = \frac{1}{E}\left[\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right] \tag{3.26}$$

$$\varepsilon_y = \frac{1}{E}\left[\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right] \tag{3.27}$$

$$\gamma_{xy} = -2\frac{(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y} \tag{3.28}$$

Substituting these expressions in equation (3.19) we obtain:

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = E\left[\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2 - \left(\frac{\partial^2\omega}{\partial x^2}\right)\left(\frac{\partial^2\omega}{\partial y^2}\right)\right] \tag{3.29}$$

An additional equation, necessary to determine ϕ and ω, is obtained by substituting expressions (3.23),(3.24) and (3.25) in the equilibrium equation (3.18), which gives

$$\frac{\partial^4\omega}{\partial x^4} + 2\frac{\partial^4\omega}{\partial x^2\partial y^2} + \frac{\partial^4\omega}{\partial y^4} = \frac{t}{D}\left[\frac{q}{t} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\omega}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2\omega}{\partial x\partial y}\right] \qquad (3.30)$$

Equations (3.29) and (3.30) are Von Karman equations. These two equations, together with the boundary conditions determine the two functions ϕ and ω. Having the stress function ϕ, we can determine the stresses in the middle surface of the plate by applying equations (3.23),(3.24) and (3.25). From the function ω, which defines the deflected surface of the plate, the bending and the shear stresses can be obtained by using the same formulas as in the case of plates with small deflection (see equations (3.15),(3.16),(3.17). Thus the investigation of large deflections of plates reduces to the solution of the two nonlinear differential equations (3.29) and (3.30).

Physically, this non-linearity arises from the increasing role played by the membrane stresses in resisting the normal force acting on the plate.

The solution of Equations (3.29) and (3.30) in the general case is unknown. Approximate or numerical solutions of the problem must then be used.

3.4 THE ENERGY METHOD

The strain energy from the membrane action, due to stretching the middle surface is given by the expression (see Timoshenko [4])

$$U_m = \frac{1}{2}\iint (N_x\varepsilon_x + N_y\varepsilon_y + N_{xy}\gamma_{xy})dxdy =$$

$$= \frac{Et}{2(1-\nu^2)}\iint[\varepsilon^2{}_x + \varepsilon^2{}_y + 2\nu\varepsilon_x\varepsilon_y + \tfrac{1}{2}(1-\nu)\gamma^2{}_{xy}]dxdy \qquad (3.31)$$

Substituting expressions (3.1), (3.2) and (3.3) for the strain components ε_x, ε_y and γ_{xy} we obtain:

$$U_m = \frac{Et}{2(1-\nu^2)}\iint[(\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial x}(\frac{\partial \omega}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + \frac{\partial v}{\partial y}(\frac{\partial \omega}{\partial y})^2 +$$

$$+ \tfrac{1}{4}\{(\frac{\partial \omega}{\partial x})^2 + (\frac{\partial \omega}{\partial y})^2\}^2 + 2\nu\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \tfrac{1}{2}\frac{\partial v}{\partial y}(\frac{\partial \omega}{\partial x})^2 +$$

$$+ \tfrac{1}{2}\frac{\partial u}{\partial x}(\frac{\partial \omega}{\partial y})^2\} + \frac{1-\nu}{2}\{(\frac{\partial u}{\partial y})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + (\frac{\partial v}{\partial x})^2 +$$

$$+ 2\frac{\partial u}{\partial y}\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}\}]dxdy \qquad (3.32)$$

Since the effect of shearing stresses on the deflections was neglected in the derivation of Equation (3.13), the corresponding expression for the strain energy due to bending contains only such terms that depend upon the actions of bending and twisting moments.

Using Equations (3.15),(3.16) and (3.17), together with the following functions for expressing the changes of curvature and twisting:

$$-\frac{\partial^2\omega}{\partial x^2}, \qquad -\frac{\partial^2\omega}{\partial y^2}, \qquad -\frac{\partial^2\omega}{\partial x\partial y}$$

The strain energy for the plate in bending can be written as

$$U_b = \frac{D}{2}\iint\{(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2})^2 - 2(1-\nu)[\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2} - (\frac{\partial^2\omega}{\partial x\partial y})^2]\}dxdy \qquad (3.33)$$

If all the edges of the plate are supported, the second term on the right-hand side of Equation (3.33) becomes zero, and the expression of strain energy in bending becomes:

$$U_b = \frac{D}{2}\iint(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2})^2\ dxdy \qquad (3.34)$$

The strain energy in the stiffeners, due to axial stress is given by:

$$U_s = \tfrac{1}{2}\oint \varepsilon_s.\sigma_s.\ A_s\ ds \qquad (3.35)$$

where the integration is evaluated along the plate boundaries, and A_s is the stiffener's area. If A_s is constant,

$$\therefore \quad U_s = \tfrac{1}{2} A_s \oint \varepsilon_s . \sigma_s . ds \tag{3.36}$$

The total strain energy U of the plate is obtained by adding the strain energy of bending, U_b, the strain energy due to strain of the middle surface, U_m, and the strain energy of the stiffeners, U_s.

The total energy (π) of the system is composed of strain energy of the plate plus the potential energy of the load. This total energy is minimum when the loaded plate is in stable equilibrium. The potential energy (V) of the load referred to the undisturbed plate level as a datum, is $V = \iint(- wq)dxdy$ and thus the total potential energy (π) is given by:

$$\pi = U_b + U_m + U_{st} + V \tag{3.37}$$

An exact solution to the plate problem:

(i) satisfies the governing Equations (3.29) and (3.30);

(ii) satisfies the boundary conditions;

(iii) minimizes the total potential energy, π, of Equation (3.37).

These three conditions are not independent of one another, as condition (i) can be derived as a consequence of condition (iii).

To apply the energy method, suitable expressions must be assumed closely approximating the displacements u, v and w. These expressions must satisfy the boundary conditions and will contain several arbitrary parameters, the magnitudes of which are determined by minimizing the total potential energy.

CHAPTER IV

DISPLACEMENT FUNCTIONS

CHAPTER IV

DISPLACEMENT FUNCTIONS

4.1 REQUIREMENTS OF THE DISPLACEMENT FUNCTIONS

A suitable displacement function should satisfy the boundary conditions and closely approximate the shape of the actual deflected surface, as the accuracy of the method depends upon how well the assumed function fits the true surface.

In the present case, the vertical deflection, and the bending moment must vanish at the boundaries, i.e.,

$$\omega = 0 = \frac{\partial^2 \omega}{\partial x^2} \quad \text{at} \quad x = \pm\frac{a}{2}$$

$$\omega = 0 = \frac{\partial^2 \omega}{\partial y^2} \quad \text{at} \quad y = \pm\frac{a}{2}$$

For a square plate with sides of length a, with the origin of the coordinate axes at the centre of the plate as shown in Fig. 4.1 and Fig.4.2, a displacement function which satisfies these boundary conditions is:

$$\omega(x,y) = \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} a_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a} \qquad (4.1)$$

where

$$m = 1,3,5, \ldots \text{ and } n = 1,3,5,\ldots$$

SQUARE PLATE WITH SIDES OF LENGTH 'a'

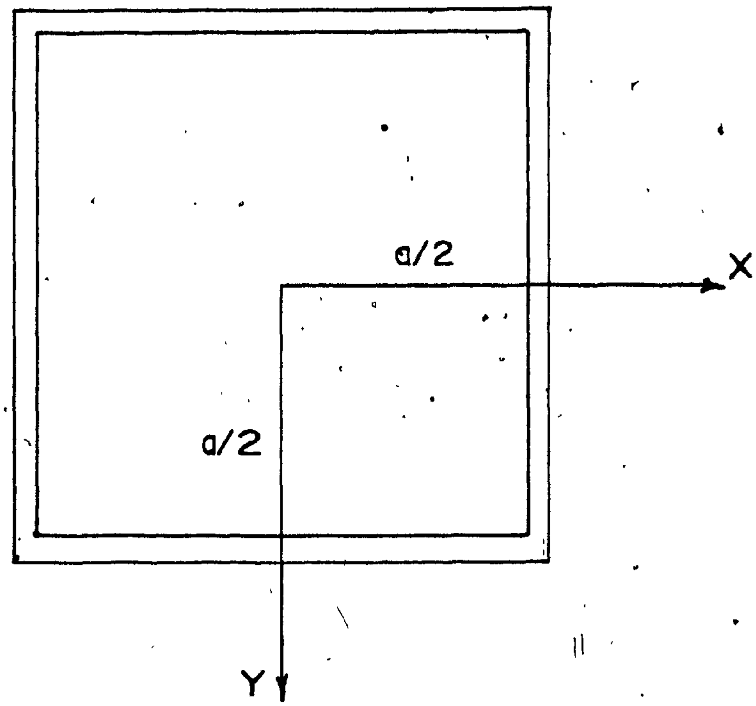


FIG (4.1)

COORDINATE AXES

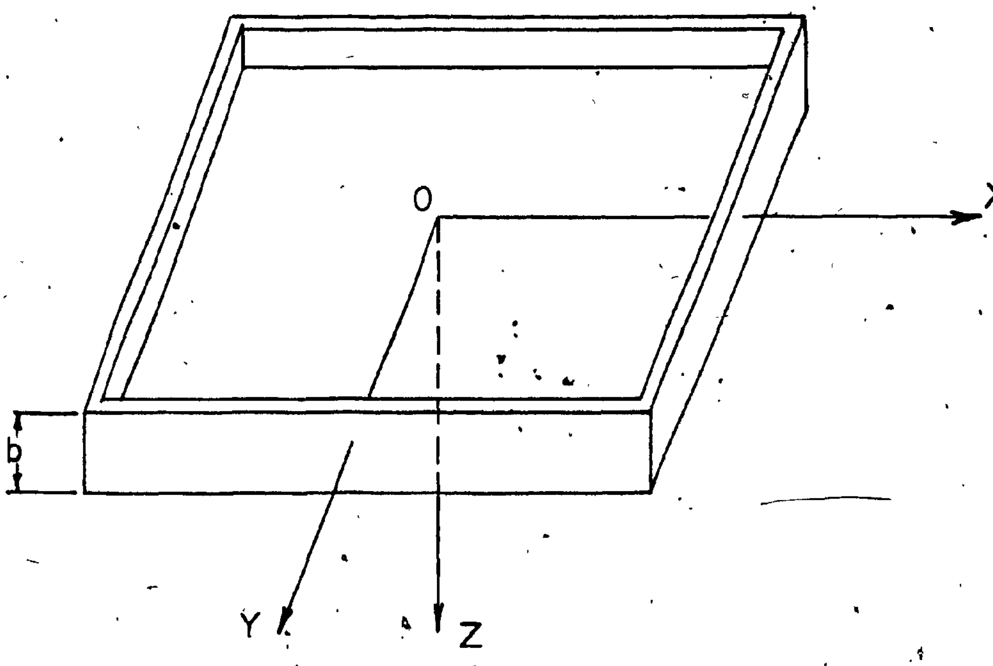


FIG (4.2)

The displacement u should, from symmetry, be an odd function of x and an even function of y. Also, the displacement v should be, again from symmetry, an odd function of y and an even function of x. Moreover, both u and v must be continuous functions of x and y, i.e.,

$$u = 0 \quad \text{at} \quad x = 0$$

and

$$v = 0 \quad \text{at} \quad y = 0$$

From the physical point of view, for the case of plates with moving boundaries, both the displacements u and v should be in some way related to the vertical displacement, ω and a suitable coupling term should be represented.

In addition to the geometric boundary conditions, no edge restraint means that the membrane stress normal to the boundaries is zero, i.e.,

$$\frac{\partial^2 \phi}{\partial y^2} = \sigma_{xm} = 0 \text{ at } x = \pm\frac{a}{2}$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = \sigma_{ym} = 0 \text{ at } y = \pm\frac{a}{2}$$

Further, the equilibrium between the stiffeners and the plate (see Fig.4.3) requires that:

$$A_s d\sigma_s - \tau_{xy_m} t\, dx = 0$$

Putting

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

gives:

$$\frac{A_s}{t}\left.\frac{\partial\sigma_s}{\partial x}\right|_{y=a/2} = -\left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{y=a/2}$$

When $A_s = 0$ gives the case of a plate without an edge stiffener, in which case

$$\left.\tau_{xy}\right|_{y=a/2} = -\left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{y=a/2} = 0$$

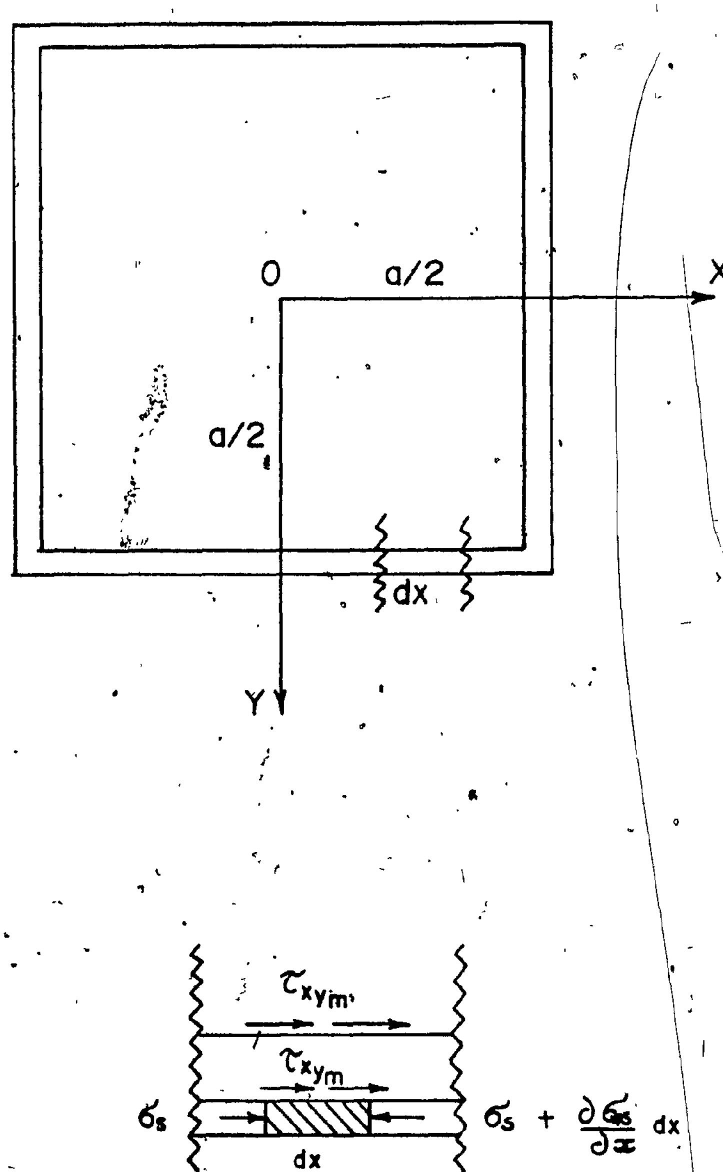
Similarly:

$$\frac{A_s}{t}\left.\frac{\partial\sigma_s}{\partial y}\right|_{x=a/2} = -\left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{x=a/2}$$

4.2 GENERATION OF THE DISPLACEMENT FUNCTIONS

Across the plate, at y = 0, the membrane stress σ_{xm} varies from a maximum tension value at the centre of the plate, to zero, at the boundaries. The tension stress, σ_{xm} at x = 0, varies with y (see Fig.4.4), the value decreasing to zero at a neutral line beyond which it becomes compression.

As there is no in-plane external force normal to the plate edges, then, at any cross-section where x = constant,



EQUILIBRIUM BETWEEN THE STIFFENER & THE PLATE

FIG (4.3)

$$\int_{-a/2}^{a/2} \sigma_x . dy + 2\sigma_s A_s = 0 \qquad (4.2)$$

and at any cross-section where y = constant:

$$\int_{-a/2}^{a/2} \sigma_y . dx + 2\sigma_s A_s = 0 \qquad (4.3)$$

The distribution of the membrane stresses can be expressed by the function:

$$\sigma_{xm} = [b_1\cos\frac{\pi x}{a} + b_3\cos\frac{3\pi x}{a} + b_5\cos\frac{5\pi x}{a} + \ldots] \\ [a_1 + a_2\cos\frac{\pi y}{a} + a_3\cos\frac{3\pi y}{a} + \ldots] \qquad (4.4)$$

In this analysis, only the first term in the first bracket and the first and the second terms in the second bracket are taken:

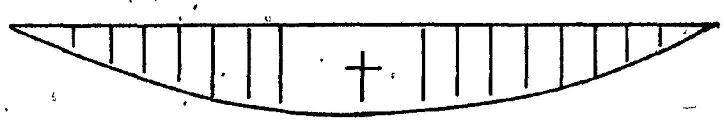
$$\therefore \quad \sigma_{xm} = (b_1\cos\frac{\pi x}{a})(a_1 + a_2\cos\frac{\pi y}{a}) \qquad (4.5)$$

Similarly

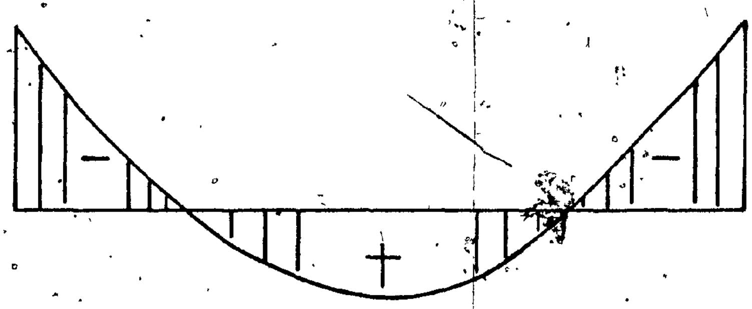
$$\sigma_{ym} = (a_1 + a_2\cos\frac{\pi x}{a})(b_1 \cos\frac{\pi y}{a}) \qquad (4.6)$$

The inplane strain is:

$$\varepsilon_x = \frac{1}{E}[\sigma_{xm} - \nu\sigma_{ym}] = \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial \omega}{\partial x})^2 \qquad (4.7)$$



σ_{x_m} at section $\bar{y}=0$



σ_{x_m} at section $x=0$

FIG(4.4)

If $\omega(x,y)$ is represented by the first term of Equation (4.1):

$$\omega = \delta_{11} \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}$$

then

$$\frac{\partial\omega}{\partial x} = -\delta_{11}\left(\frac{\pi}{a}\right) \sin\frac{\pi x}{a} \cos\frac{\pi y}{a}$$

$$\frac{1}{2}\left(\frac{\partial\omega}{\partial x}\right)^2 = \frac{\delta_{11}^2}{2}\left(\frac{\pi^2}{a^2}\right) \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a} \qquad (4.8)$$

From Equations (4.7) and (4.8)

$$\frac{\partial u}{\partial x} = \frac{1}{E}\left(b_1\cos\frac{\pi x}{a}\right)\left(a_1 + a_2\cos\frac{\pi y}{a}\right) - \frac{\nu}{E}\left(a_1 + a_2\cos\frac{\pi x}{a}\right)\left(b_1\cos\frac{\pi y}{a}\right)$$

$$- \frac{1}{2}\left(\frac{\pi}{a}\right)^2 \delta_{11}^2 \sin^2\frac{\pi x}{a} \cos^2\frac{\pi y}{a}$$

Integrating the above equation,

$$u = \frac{1}{E}\left(b_1\sin\frac{\pi x}{a}\right)\left(a_1 + a_2\cos\frac{\pi y}{a}\right) - \frac{\nu}{E}\left[a_1x + a_2\left(\frac{a}{\pi}\right)\sin\frac{\pi x}{a}\right].$$

$$.\left[b_1\cos\frac{\pi y}{a}\right] - \frac{1}{2}\left(\frac{\pi}{a}\right)^2 \delta_{11}^2\left[\frac{x}{2} - \frac{\sin\frac{2\pi x}{a}}{4}\right]\left[\cos^2\frac{\pi y}{a}\right] + f_1(y)$$

From symmetry and continuity of the function $u(x,y)$

at $x = 0$, $u = 0$ $\quad\therefore f_1(y) = 0$

$$\therefore \quad u = \frac{1}{E} b_1 a_1 \sin\frac{\pi x}{a} + \frac{1}{E} a_1 a_2 \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} - \frac{\nu}{E} a_1 b_1 x \cos\frac{\pi y}{a}$$

$$- \frac{\nu}{E} a_2 \left(\frac{a}{\pi}\right) b_1 \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} - \frac{1}{4}\left(\frac{\pi}{a}\right)^2 \delta_{11}^2 \, x \cos^2\frac{\pi y}{a}$$

$$+ \frac{1}{8}\frac{\pi^2}{a^2} \delta_{11}^2 \sin\frac{2\pi x}{a} \cos^2\frac{\pi y}{a}$$

$$u = C_1 \sin\frac{\pi x}{a} + C_3 \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} + \left(\frac{\pi x}{a}\right) C_2 \cos\frac{\pi y}{a} +$$

$$+ \left(\frac{\pi x}{a}\right) C_4 \cos^2\frac{\pi y}{a} + C_5 \cos^2\frac{\pi y}{a} \sin\frac{2\pi x}{a} \qquad (4.9)$$

where

$$\begin{aligned} C_2 &= -\nu C_1 \\ C_4 &= -\frac{1}{4}\left(\frac{\pi}{a}\right)\delta_{11}^2 \\ C_5 &= \frac{1}{8}\left(\frac{\pi}{a}\right)\delta_{11}^2 \end{aligned} \qquad (4.10)$$

Interchanging x with y, we can express v as:

$$v = C_1 \sin\frac{\pi y}{a} + C_3 \sin\frac{\pi y}{a} \cos\frac{\pi x}{a} + C_2 \left(\frac{\pi y}{a}\right) \cos\frac{\pi x}{a} +$$

$$+ C_4 \left(\frac{\pi y}{a}\right) \cos^2\frac{\pi x}{a} + C_5 \cos^2\frac{\pi x}{a} \sin\frac{2\pi y}{a} \qquad (4.11)$$

and

$$\omega = \delta_{11} \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} \qquad (4.12)$$

If ω is expressed by four terms of Equation (4.1):

$$\omega = \delta_{11} \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_{13} \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a} + \delta_{31} \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} +$$

$$+ \delta_{33} \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a}$$

For a square plate $\delta_{13} = \delta_{31}$

$$\omega = \delta_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a} + \delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} +$$

$$+ \delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a} \qquad (4.13)$$

where:

$$\delta_1 = \delta_{11}, \qquad \delta_2 = \delta_{13} = \delta_{31}, \qquad \delta_3 = \delta_{33}$$

$$\varepsilon_x = \frac{1}{E}[\sigma_{xm} - \nu\sigma_{ym}] \qquad (4.7)$$

$$\varepsilon_x = \frac{1}{E}[a_1 b_1 \cos\frac{\pi x}{a} + a_2 b_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}$$

$$- \nu a_1 b_1 \cos\frac{\pi y}{a} - \nu a_2 b_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}]$$

$$\varepsilon_x = B_1 \cos\frac{\pi x}{a} + B_2 \cos\frac{\pi y}{a} + B_3 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}$$

$$(4.14)$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} = {} & B_1 \cos\frac{\pi x}{a} + B_2 \cos\frac{\pi y}{a} + B_3 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} \\
& - \frac{\pi^2}{2a^2}\Big[\delta_1^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a} + \delta_2^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{3\pi y}{a} \\
& + 9\delta_2^2 \sin^2 \frac{3\pi x}{a} \cos^2 \frac{\pi y}{a} + 9\delta_3^2 \sin^2 \frac{3\pi x}{a} \cos^2 \frac{3\pi y}{a} + \\
& + 2\delta_1\delta_2 \sin^2 \frac{\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + 6\delta_1\delta_2 \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a} \cos^2 \frac{\pi y}{a} + \\
& + 6\delta_1\delta_3 \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + 6\delta_2^2 \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a} \cos\frac{\pi y}{a} \cos\frac{3\pi y}{a} + \\
& + 6\delta_2\delta_3 \sin \frac{\pi x}{a} \sin \frac{3\pi x}{a} \cos^2 \frac{3\pi y}{a} + \\
& + 18\delta_2\delta_3 \sin^2 \frac{3\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a}\Big]
\end{aligned}
$$

$$u = \frac{a}{\pi}[B_1 \sin \frac{\pi x}{a} + B_2 (\frac{\pi x}{a}) \cos \frac{\pi y}{a} + B_3 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a}]$$

$$- \frac{\pi}{2a}[\delta_1^2 (\frac{\pi x}{2a} - \frac{1}{4} \sin \frac{2\pi x}{a}) \cos^2 \frac{\pi y}{a}$$

$$+ \delta_2^2 (\frac{\pi x}{2a} - \frac{1}{4} \sin \frac{2\pi x}{a}) \cos^2 \frac{3\pi y}{a}] - \frac{\pi}{2a} [9\delta_2^2 (\frac{\pi x}{2a} -$$

$$- \frac{1}{12} \sin \frac{6\pi x}{a}) \cos^2 \frac{\pi y}{a} + 9\delta_3^2 (\frac{\pi x}{2a} - \frac{1}{12} \sin \frac{6\pi x}{a}) \cos^2 \frac{3\pi y}{a} +$$

$$+ 2\delta_1 \delta_2 (\frac{\pi x}{2a} - \frac{1}{4} \sin \frac{2\pi x}{a}) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} -$$

$$- \frac{\pi}{2a}[6\delta_1 \delta_2 (\frac{1}{4} \sin \frac{2\pi x}{a} - \frac{1}{8} \sin \frac{4\pi x}{a}) \cos^2 \frac{\pi y}{a} +$$

$$+ 6\delta_1 \delta_3 (\frac{1}{4} \sin \frac{2\pi x}{a} - \frac{1}{8} \sin \frac{4\pi x}{a}) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} +$$

$$+ 6\delta_2^2 (\frac{1}{4} \sin \frac{2\pi x}{a} - \frac{1}{8} \sin \frac{4\pi x}{a}) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a}]$$

$$- \frac{\pi}{2a}[6\delta_2 \delta_3 (\frac{1}{4} \sin \frac{2\pi x}{a} - \frac{1}{8} \sin \frac{4\pi x}{a}) \cos^2 \frac{3\pi y}{a} +$$

$$+ 18\delta_2 \delta_3 (\frac{\pi x}{2a} - \frac{1}{12} \sin \frac{6\pi x}{a}) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a}]$$

$$
\begin{aligned}
u = {} & C_1 \sin \frac{\pi x}{a} + C_2 \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} + C_3 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} + \\
& + C_4 \left(\frac{\pi x}{a}\right) \cos^2 \frac{\pi y}{a} + C_5 \sin \frac{2\pi x}{2} \cos^2 \frac{\pi y}{a} + \\
& + C_6 \left(\frac{\pi x}{a}\right) \cos^2 \frac{3\pi y}{a} + C_7 \sin \frac{2\pi x}{2} \cos^2 \frac{3\pi y}{a} + \\
& + C_8 \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_9 \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + C_{10} \left(\frac{\pi x}{a}\right) \cos^2 \frac{\pi y}{a} + C_{11} \sin \frac{6\pi x}{a} \cos^2 \frac{\pi y}{a} + \\
& + C_{12} \sin \frac{2\pi x}{a} \cos^2 \frac{\pi y}{a} + C_{13} \sin \frac{4\pi x}{a} \cos^2 \frac{\pi y}{a} + \\
& + C_{14} \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + C_{15} \sin \frac{4\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_{16} \left(\frac{\pi x}{a}\right) \cos^2 \frac{3\pi y}{a} + \\
& + C_{17} \sin \frac{6\pi x}{a} \cos^2 \frac{3\pi y}{a} + C_{18} \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + C_{19} \sin \frac{4\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_{20} \sin \frac{2\pi x}{a} \cos^2 \frac{3\pi y}{a} + \\
& + C_{21} \sin \frac{4\pi x}{a} \cos^2 \frac{3\pi y}{a} + C_{22} \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + \\
& + C_{23} \sin \frac{6\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a}
\end{aligned}
\tag{4.15}
$$

Similarly, by interchanging x with y

$$v = C_1 \sin \frac{\pi y}{a} + C_2 \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} + C_3 \sin \frac{\pi y}{a} \cos \frac{\pi x}{a} +$$

$$+ C_4 \left(\frac{\pi y}{a}\right) \cos^2 \frac{\pi x}{a} + C_5 \sin \frac{2\pi y}{a} \cos^2 \frac{\pi x}{a} +$$

$$+ C_6 \left(\frac{\pi y}{a}\right) \cos^2 \frac{3\pi x}{a} + C_7 \sin \frac{2\pi y}{a} \cos^2 \frac{3\pi x}{a} +$$

$$+ C_8 \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_9 \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} +$$

$$+ C_{10} \left(\frac{\pi y}{a}\right) \cos^2 \frac{\pi x}{a} + C_{11} \sin \frac{6\pi y}{a} \cos^2 \frac{\pi x}{a} +$$

$$+ C_{12} \sin \frac{2\pi y}{a} \cos^2 \frac{\pi x}{a} + C_{13} \sin \frac{4\pi y}{a} \cos^2 \frac{\pi x}{a} +$$

$$+ C_{14} \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} +$$

$$+ C_{15} \sin \frac{4\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_{16} \left(\frac{\pi y}{a}\right) \cos^2 \frac{3\pi x}{a} +$$

$$+ C_{17} \sin \frac{6\pi y}{a} \cos^2 \frac{3\pi x}{a} + C_{18} \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} +$$

$$+ C_{19} \sin \frac{4\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_{20} \sin \frac{2\pi y}{a} \cos^2 \frac{3\pi x}{a} +$$

$$+ C_{21} \sin \frac{4\pi y}{a} \cos^2 \frac{3\pi x}{a} + C_{22} \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} +$$

$$+ C_{23} \sin \frac{6\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a}$$

$$(4.16)$$

and

$$\omega = \delta_1 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a} +$$

$$+ \delta_2 \cos\frac{3\pi x}{a}\cos\frac{\pi y}{a} + \delta_3 \cos\frac{3\pi x}{a}\cos\frac{3\pi y}{a} \qquad (4.13)$$

where

$$C_2 = -\nu C_1$$

$$C_4 = -\frac{\pi}{4a}\delta_1^2 \qquad C_5 = \frac{\pi}{8a}\delta_1^2$$

$$C_6 = -\frac{\pi}{4a}\delta_2^2 \qquad C_7 = \frac{\pi}{8a}\delta_2^2$$

$$C_8 = -\frac{\pi}{2a}\delta_1\delta_2 \qquad C_9 = \frac{\pi}{4a}\delta_1\delta_2$$

$$C_{10} = -\frac{9}{4}\left(\frac{\pi}{a}\right)\delta_2^2 \qquad C_{11} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_2^2$$

$$C_{12} = -\frac{3}{4}\left(\frac{\pi}{a}\right)\delta_1\delta_2 \qquad C_{13} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_1\delta_2$$

$$C_{14} = -\frac{3}{4}\left(\frac{\pi}{a}\right)\delta_2^2 \qquad C_{15} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_2^2 \qquad (4.17)$$

$$C_{16} = -\frac{9}{4}\left(\frac{\pi}{a}\right)\delta_3^2 \qquad C_{17} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_3^2$$

$$C_{18} = -\frac{3}{4}\left(\frac{\pi}{a}\right)\delta_1\delta_3 \qquad C_{19} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_1\delta_3$$

$$C_{20} = -\frac{3}{4}\left(\frac{\pi}{a}\right)\delta_2\delta_3 \qquad C_{21} = \frac{3}{8}\left(\frac{\pi}{a}\right)\delta_2\delta_3$$

$$C_{22} = -\frac{9}{2}\left(\frac{\pi}{a}\right)\delta_2\delta_3 \qquad C_{23} = \frac{3}{4}\left(\frac{\pi}{a}\right)\delta_2\delta_3$$

From Equation (3.6)

$$\sigma_{ym} = \frac{\partial^2\phi}{\partial x^2} = (a_1 + a_2 \cdot \cos\frac{\pi x}{a})(b_1 \cos\frac{\pi y}{a})$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = [a_1 x + a_2 (\frac{a}{\pi}) \sin\frac{\pi x}{a}](b_1 \cos\frac{\pi y}{a}) +$$

$$+ f_2(y) \qquad (4.18)$$

from symmetry and continuity of the function $\phi = \phi(x,y)$

$$\text{at} \quad x = 0 \qquad \frac{\partial\phi}{\partial x} = 0$$

$$\therefore \quad f_2(y) = 0$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = [a_1 x + a_2 (\frac{a}{\pi}) \sin\frac{\pi x}{a}][b_1 \cos\frac{\pi y}{a}]$$

$$\phi(x,y) = [\frac{a_1 x^2}{2} - a_2 (\frac{a}{\pi})^2 \cos\frac{\pi x}{a}][b_1 \cos\frac{\pi y}{a}] +$$

$$+ f_3(y) \qquad (4.19)$$

$$\omega = \delta_1 \cos\frac{\pi y}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a}$$

$$+ \delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} + \delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a}$$

$$(4.13)$$

4.3 REMARKS

(1) Equations (3.19) and (3.13) for the stress function do not satisfy the governing differential equations of the problem (3.29) and (3.30), thus the solution is an approximate one.

(2) Although high accuracy for the deflection of plates can be attained by the minimization of the total potential energy, the accuracy of the internal forces is usually somewhat reduced because first, second or third-order derivatives of u,v, and ω are involved in their determination.

(3) For non-stiffened plates, the natural boundary condition of zero shear stress at the edges should be satisfied.; i.e.,

$$\text{at} \quad x = \pm \frac{a}{2} \qquad \tau_{xym} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Therefore:

$$\frac{\partial \tau_{xym}}{\partial y} = 0 \quad \text{at} \quad x = \pm \frac{a}{2} \tag{4.20}$$

Substituting the condition of Equation (4.20) into the equilibrium Equation (3.4), we obtain

$$\frac{\partial \sigma_{xm}}{\partial x} = 0 \quad \text{at} \quad x = \pm \frac{a}{2} \tag{4.21}$$

This condition implies that the σ_{xm} distribution should have zero slope at $x = \pm \frac{a}{2}$.

Unfortunately, the condition of Equation (4.21) is not satisfied by the assumption of σ_{xm} (Equation 4.5), and this puts another restriction of the use of this displacement function for the case of plates with edge stiffeners.

In practice, the accuracy of the assumption (4.5) increases as the stiffener area increases, and this assumption yields accurate results when the stiffener area tends to infinity.

CHAPTER V

SOLUTION

CHAPTER V

SOLUTION

5.1 INTRODUCTION

To obtain an approximate solution of the problem of simply supported plates with stiffeners around the boundaries, the energy method is used.

The total strain energy U of the plate is obtained by adding the strain energy due to bending U_b, expression (3.34), the strain energy due to strain of the middle surface U_m, expression (3.32) and the strain energy due to compression in the stiffeners, U_s, expression (3.36).

The principle of minimization of the total potential energy π then gives

$$\delta\pi = \delta U + \delta V = 0 \tag{5.1}$$

which holds for any variation of the displacements u,v, and ω.

By deriving the variation of U, we can obtain from Equation (5.1), the system of equations (3.29) and (3.30), only if we have the correct displacements u,v, and ω.

To obtain an approximate solution to the problem, the assumed u,v and ω are given by the expressions(4.15),(4.16) and (4.13), derived in Chapter IV, given below:

$$
\begin{aligned}
u = {} & C_1 \sin \frac{\pi x}{a} + C_2 \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} + C_3 \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \\
& + C_4 \left(\frac{\pi x}{a}\right) \cos^2 \frac{\pi y}{a} + C_5 \sin \frac{2\pi x}{2} \cos^2 \frac{\pi y}{a} \\
& + C_6 \left(\frac{\pi x}{a}\right) \cos^2 \frac{3\pi y}{a} + C_7 \sin \frac{2\pi x}{2} \cos^2 \frac{3\pi y}{a} \\
& + C_8 \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_9 \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} \\
& + C_{10} \left(\frac{\pi x}{a}\right) \cos^2 \frac{\pi y}{a} + C_{11} \sin \frac{6\pi x}{a} \cos^2 \frac{\pi y}{a} \\
& + C_{12} \sin \frac{2\pi x}{a} \cos^2 \frac{\pi y}{a} + C_{13} \sin \frac{4\pi x}{a} \cos^2 \frac{\pi y}{a} \\
& + C_{14} \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} \\
& + C_{15} \sin \frac{4\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_{16} \left(\frac{\pi x}{a}\right) \cos^2 \frac{3\pi y}{a} \\
& + C_{17} \sin \frac{6\pi x}{a} \cos^2 \frac{3\pi y}{a} + C_{18} \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} \\
& + C_{19} \sin \frac{4\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} + C_{20} \sin \frac{2\pi x}{a} \cos^2 \frac{3\pi y}{a} \\
& + C_{21} \sin \frac{4\pi x}{a} \cos^2 \frac{3\pi y}{a} + C_{22} \left(\frac{\pi x}{a}\right) \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a} \\
& + C_{23} \sin \frac{6\pi x}{a} \cos \frac{\pi y}{a} \cos \frac{3\pi y}{a}
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
v = {} & C_1 \sin \frac{\pi y}{a} + C_2 \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} + C_3 \sin \frac{\pi y}{a} \cos \frac{\pi x}{a} \\
& + C_4 \left(\frac{\pi y}{a}\right) \cos^2 \frac{\pi x}{a} + C_5 \sin \frac{2\pi y}{a} \cos^2 \frac{\pi x}{a} \\
& + C_6 \left(\frac{\pi y}{a}\right) \cos^2 \frac{3\pi x}{a} + C_7 \sin \frac{2\pi y}{a} \cos^2 \frac{3\pi x}{a} \\
& + C_8 \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_9 \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} \\
& + C_{10} \left(\frac{\pi y}{a}\right) \cos^2 \frac{\pi x}{a} + C_{11} \sin \frac{6\pi y}{a} \cos^2 \frac{\pi x}{a} \\
& + C_{12} \sin \frac{2\pi y}{a} \cos^2 \frac{\pi x}{a} + C_{13} \sin \frac{4\pi y}{a} \cos^2 \frac{\pi x}{a} \\
& + C_{14} \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} \\
& + C_{15} \sin \frac{4\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_{16} \left(\frac{\pi y}{a}\right) \cos^2 \frac{3\pi x}{a} \\
& + C_{17} \sin \frac{6\pi y}{a} \cos^2 \frac{3\pi x}{a} + C_{18} \sin \frac{2\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} \\
& + C_{19} \sin \frac{4\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} + C_{20} \sin \frac{2\pi y}{a} \cos^2 \frac{3\pi x}{a} \\
& + C_{21} \sin \frac{4\pi y}{a} \cos^2 \frac{3\pi x}{a} + C_{22} \left(\frac{\pi y}{a}\right) \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a} \\
& + C_{23} \sin \frac{6\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a}
\end{aligned}
\tag{4.16}
$$

$$\omega = \delta_1 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} + \delta_2 \cos \frac{\pi x}{a} \cos \frac{3\pi y}{a}$$

$$+ \delta_2 \cos \frac{3\pi x}{a} \cos \frac{\pi y}{a} + \delta_3 \cos \frac{3\pi x}{a} \cos \frac{3\pi y}{a} \quad (4.13)$$

These expressions contain 26 parameters $C_1, C_2, \ldots, \delta_3$, which will be determined from Equation (5.1), which must be satisfied for any variation of each of these parameters.

5.2 U_m, STRAIN ENERGY DUE TO MEMBRANE ACTION

The strain energy due to the strain of the middle surface can be evaluated using Equation (3.31)

$$U_m = \frac{Et}{2(1-\nu^2)} \iint [\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + (\frac{1-\nu}{2})\, \gamma_{xy}^2\, dxdy$$

Substituting

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y} \quad (3.3)$$

$$\therefore \quad U_m = \frac{Et}{2(1-\nu^2)} \iint [(\varepsilon_x^2+\varepsilon_y^2) + 2\nu\varepsilon_x\varepsilon_y + (\frac{1-\nu}{2})(\frac{\partial\omega}{\partial x})^2(\frac{\partial\omega}{\partial y})^2$$

$$+ (\frac{1-\nu}{2})\{(\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2\} + (\frac{1-\nu}{2})\{2(\frac{\partial u}{\partial y})(\frac{\partial v}{\partial x})$$

$$+ 2(\frac{\partial v}{\partial x})(\frac{\partial \omega}{\partial x})(\frac{\partial \omega}{\partial y})\}]dxdy \quad (5.2)$$

Since the loading and deformations are assumed to be symmetrical only one-quarter of the plate need to be considered. The domain of the problem is therefore $0 \leq x \leq \frac{a}{2}$, $0 \leq y \leq \frac{a}{2}$.

We have from Equation (4.14)

$$\varepsilon_x = \frac{\pi}{a} C_1 \cos \frac{\pi x}{a} + \frac{\pi}{a} C_2 \cos \frac{\pi y}{a} + \frac{\pi}{a} C_3 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \tag{4.14}$$

$$\varepsilon_x^2 = \left(\frac{\pi}{a}\right)^2 \Big[C_1^2 \cos^2 \frac{\pi x}{a} + C_2^2 \cos^2 \frac{\pi y}{a} + C_3^2 \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a}$$

$$+ 2C_1C_3 \cos^2 \frac{\pi x}{a} \cos \frac{\pi y}{a} + 2C_1C_2 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}$$

$$+ 2C_2C_3 \cos \frac{\pi x}{a} \cos^2 \frac{\pi y}{a}\Big]$$

Using the integration table in Appendix A

$$\int_0^{a/2} \int_0^{a/2} \varepsilon_x^2 \, dxdy = \frac{\pi^2}{8} C_1^2 + \frac{\pi^2}{16} C_3^2 + \frac{\pi^2}{8} C_2^2$$

$$+ \frac{\pi}{2} C_1C_3 + 2C_1C_2 + \frac{\pi}{2} C_2C_3$$

For a square plate, one can prove that

$$\int_0^{a/2} \int_0^{a/2} \varepsilon_x^2 \, dxdy = \int_0^{a/2} \int_0^{a/2} \varepsilon_y^2 \, dxdy$$

$$\therefore \int_0^{a/2}\int_0^{a/2} (\varepsilon_x^2+\varepsilon_y^2)\,dxdy = \frac{\pi^2}{4} C_1^2 + \frac{\pi^2}{4} C_2^2 + \frac{\pi^2}{8} C_3^2$$

$$+ 4C_1C_2 + \pi C_1C_3 + \pi C_2C_3 \qquad (5.3)$$

Interchanging x and y in Equation (4.14)

$$\therefore \varepsilon_y = \frac{\pi}{a} C_1 \cos \frac{\pi y}{a} + \frac{\pi}{a} C_2 \cos \frac{\pi x}{a} + \frac{\pi}{a} C_3 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}$$

$$\varepsilon_x\varepsilon_y = \frac{\pi^2}{a^2} [C_1^2 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} + C_1C_3 \cos \frac{\pi x}{a} \cos^2 \frac{\pi y}{a}$$

$$+ C_1C_2 \cos^2 \frac{\pi y}{a} + C_1C_3 \cos^2 \frac{\pi x}{a} \cos \frac{\pi y}{a}$$

$$+ C_3^2 \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a} + C_2C_3 \cos \frac{\pi x}{a} \cos^2 \frac{\pi y}{a}$$

$$+ C_1C_2 \cos^2 \frac{\pi x}{a} + C_2C_3 \cos^2 \frac{\pi x}{a} \cos \frac{\pi y}{a}$$

$$+ C_2^2 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a}]$$

Using the integration table in Appendix A

$$\int_0^{a/2}\int_0^{a/2} \varepsilon_x\varepsilon_y\, d_x d_y = C_1^2 + \frac{\pi^2}{16} C_3^2 + C_2^2 + \frac{\pi}{2} C_1C_3$$

$$+ \frac{\pi^2}{4} C_1C_2 + \frac{\pi}{2} C_2C_3$$

For $\nu = .316$

$$2\nu \int_0^{a/2} \int_0^{a/2} \varepsilon_x \varepsilon_y \, dxdy = .632\left(C_1^2 + C_2^2 + \frac{\pi^2}{16} C_3^2 + \frac{\pi^2}{4} C_1C_2 + \frac{\pi}{2} C_1C_3 + \frac{\pi}{2} C_2C_3\right) \tag{5.4}$$

Using Equation (4.13) gives:

$$\frac{\partial\omega}{\partial x} = -\frac{\pi}{a}\left[\delta_1 \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \sin\frac{\pi x}{a} \cos\frac{3\pi y}{a} + 3\delta_2 \sin\frac{3\pi x}{a} \cos\frac{\pi y}{a} + 3\delta_3 \sin\frac{3\pi x}{a} \cos\frac{3\pi y}{a}\right]$$

$$\frac{\partial\omega}{\partial y} = -\frac{\pi}{a}\left[\delta_1 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a} + 3\delta_2 \cos\frac{\pi x}{a} \sin\frac{3\pi y}{a} + \delta_2 \cos\frac{3\pi x}{a} \sin\frac{\pi y}{a} + 3\delta_3 \cos\frac{3\pi x}{a} \sin\frac{3\pi y}{a}\right]$$

Using the trigonometric identity

$$\sin\alpha \cos\beta = \frac{[\sin(\alpha+\beta) + \sin(\alpha-\beta)]}{2}$$

$$\cos\alpha \sin\beta = \frac{[\sin(\alpha+\beta) - \sin(\alpha-\beta)]}{2}$$

where:

$$\alpha > \beta$$

$$
\begin{aligned}
\frac{\partial \omega}{\partial x}\frac{\partial \omega'}{\partial y} = \frac{\pi^2}{4a} [& \delta_1^2 \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{a} + \delta_1\delta_2 \sin\frac{2\pi x}{a} (\sin\frac{4\pi y}{a} \\
& - \sin\frac{2\pi y}{a}) + 3\delta_1\delta_2 (\sin\frac{4\pi x}{a} + \sin\frac{2\pi x}{a}) \sin\frac{2\pi y}{a} \\
& + 3\delta_1\delta_3 (\sin\frac{4\pi x}{a} + \sin\frac{2\pi x}{a}) (\sin\frac{4\pi y}{a} - \sin\frac{2\pi y}{a}) \\
& + 3\delta_1\delta_2 \sin\frac{2\pi x}{a} (\sin\frac{4\pi y}{a} + \sin\frac{2\pi y}{a}) \\
& + 3\delta_2^2 \sin\frac{2\pi x}{a} \sin\frac{6\pi y}{a} + 9\delta_2^2 (\sin\frac{4\pi x}{a} \\
& + \sin\frac{2\pi x}{a}) (\sin\frac{4\pi y}{a} + \sin\frac{2\pi y}{a}) + 9\delta_2\delta_3 (\sin\frac{4\pi x}{a} \\
& + \sin\frac{2\pi x}{a}) (\sin\frac{6\pi y}{a}) + \delta_1\delta_2 (\sin\frac{4\pi x}{a} - \\
& - \sin\frac{2\pi x}{a}) (\sin\frac{2\pi y}{a}) + \delta_2^2 (\sin\frac{4\pi x}{a} - \sin\frac{2\pi x}{a}) (\sin\frac{4\pi y}{a} \\
& - \sin\frac{2\pi y}{a}) + 3\delta_2^2 \sin\frac{6\pi x}{a} \sin\frac{2\pi y}{a} \\
& + 3\delta_2\delta_3 \sin\frac{6\pi x}{a} (\sin\frac{4\pi y}{a} - \sin\frac{2\pi y}{a}) + 3\delta_1\delta_3 (\sin\frac{4\pi x}{a} \\
& - \sin\frac{2\pi x}{a}) (\sin\frac{4\pi y}{a} + \sin\frac{2\pi y}{a}) + 2\delta_2\delta_3 (\sin\frac{4\pi x}{a} \\
& - \sin\frac{2\pi x}{a}) (\sin\frac{6\pi y}{a}) + 9\delta_2\delta_3 \sin\frac{6\pi x}{a} (\sin\frac{4\pi y}{a} \\
& + \sin\frac{2\pi y}{a}) + 9\delta_3^2 \sin\frac{6\pi x}{a} \sin\frac{6\pi y}{a}].
\end{aligned}
$$

$$\therefore \quad \frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y} = \frac{\pi^2}{4a^2}[D_1 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} + D_2 \sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a}$$

$$+ D_2 \sin\frac{4\pi x}{a}\sin\frac{2\pi y}{a} + D_3 \sin\frac{4\pi x}{a}\sin\frac{4\pi y}{a}$$

$$+ D_4 \sin\frac{2\pi x}{a}\sin\frac{6\pi y}{a} + D_4 \sin\frac{6\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ D_5 \sin\frac{4\pi x}{a}\sin\frac{6\pi y}{a} + D_5 \sin\frac{6\pi x}{a}\sin\frac{4\pi y}{a}$$

$$+ D_6 \sin\frac{6\pi x}{a}\sin\frac{6\pi y}{a}] \qquad (5.5)$$

where

$$D_1 = \delta_1^2 + 10\delta_2^2 + 4\delta_1\delta_2 - 6\delta_1\delta_3$$

$$D_2 = 4\delta_1\delta_2 + 8\delta_2^2$$

$$D_3 = 10\delta_2^2 + 6\delta_1\delta_3$$

$$D_4 = 3\delta_2^2 + 6\delta_2\delta_3$$

$$D_5 = 12\delta_2\delta_3$$

$$D_6 = 9\delta_3^2 \qquad (5.6)$$

$$\int_0^{a/2}\int_0^{a/2} \left(\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}\right)^2 dxdy = \frac{\pi^4}{256a^4}[D_1^2 + D_2^2 + D_2^2 + D_3^2 + D_4^2 +$$

$$+ D_4^2 + D_5^2 + D_5^2 + D_6^2]$$

$$= \frac{\pi^4}{256a^4}[D_1^2 + 2D_2^2 + D_3^2 + 2D_4^2 + 2D_5^2 + D_6^2]$$

For $\qquad \nu = .316$

$$\therefore \quad \frac{1-\nu}{2} \int_0^{a/2} \int_0^{a/2} \left(\frac{\partial\omega}{\partial x} \frac{\partial\omega}{\partial y}\right)^2 dxdy = .342 \left[\frac{\pi^4}{a^2}\right]\left[\frac{\delta_1^4}{256} + \frac{1}{32}\delta_1^3\delta_2\right.$$

$$+ \frac{17}{64}\delta_1^2\delta_2^2 + \frac{13}{16}\delta_1\delta_2^3 + \frac{173}{128}\delta_2^4 + \frac{81}{256}\delta_3^4 - \frac{6}{128}\delta_1^3\delta_3$$

$$\left. + \frac{9}{32}\delta_1^2\delta_3^2 - \frac{3}{16}\delta_1^2\delta_2\delta_3 + \frac{45}{32}\delta_2^2\delta_3^2 + \frac{9}{32}\delta_2^3\delta_3\right] \tag{5.7}$$

$$\frac{\partial u}{\partial y} = -\frac{\pi}{a}\Big[C_2\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{a} + C_3\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} + C_4\left(\frac{\pi x}{a}\right)\sin\frac{2\pi y}{a}$$

$$+ C_5\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} + 3C_6\left(\frac{\pi x}{a}\right)\sin\frac{6\pi y}{a}$$

$$+ 3C_7\sin\frac{2\pi x}{a}\sin\frac{6\pi y}{a} + C_8\left(\frac{\pi x}{a}\right)\sin\frac{2\pi y}{a} + 2C_8\left(\frac{\pi x}{a}\right)\sin\frac{4\pi y}{a}$$

$$+ C_9\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} + 2C_9\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a}$$

$$+ C_{10}\left(\frac{\pi x}{a}\right)\sin\frac{2\pi y}{a} + C_{11}\sin\frac{6\pi x}{a}\sin\frac{2\pi y}{a} + C_{12}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ C_{13}\sin\frac{4\pi x}{a}\sin\frac{2\pi y}{a} + C_{14}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ 2C_{14}\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a} + C_{15}\sin\frac{4\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ 2C_{15}\sin\frac{4\pi x}{a}\sin\frac{4\pi y}{a} + 3C_{16}\left(\frac{\pi x}{a}\right)\sin\frac{6\pi y}{a}$$

$$+ 3C_{17}\sin\frac{6\pi x}{a}\sin\frac{6\pi y}{a} + C_{18}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ 2C_{18}\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a} + C_{19}\sin\frac{4\pi x}{a}\sin\frac{2\pi y}{a}$$

$$+ 2C_{19}\sin\frac{4\pi x}{a}\ \sin\frac{4\pi y}{a} + 3C_{20}\sin\frac{2\pi x}{a}\ \sin\frac{6\pi y}{a}$$

$$+ 3C_{21}\sin\frac{4\pi x}{a}\ \sin\frac{6\pi y}{a} + C_{22}\left(\frac{\pi x}{a}\right)\sin\frac{2\pi y}{a}$$

$$+ 2C_{22}\left(\frac{\pi x}{a}\right)\sin\frac{4\pi y}{a} + C_{23}\sin\frac{6\pi x}{a}\ \sin\frac{2\pi y}{a}$$

$$+ 2C_{23}\sin\frac{6\pi x}{a}\ \sin\frac{4\pi y}{a}$$

Simplifying:

$$\frac{\partial u}{\partial y} = -\frac{\pi}{a}\Big[C_2\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{a} + C_3\ \sin\frac{\pi x}{a}\ \sin\frac{\pi y}{a}$$

$$+ K_1\left(\frac{\pi x}{a}\right)\sin\ \frac{2\pi y}{a} + K_2\ \sin\ \frac{2\pi x}{a}\ \sin\ \frac{2\pi y}{a}$$

$$+ K_6\left(\frac{\pi x}{a}\right)\sin\ \frac{6\pi y}{a} + K_7\ \sin\frac{2\pi x}{a}\ \sin\ \frac{6\pi y}{a}$$

$$+ K_8\left(\frac{\pi x}{a}\right)\sin\ \frac{4\pi y}{a} + K_3\ \sin\ \frac{2\pi x}{a}\ \sin\frac{4\pi y}{a}$$

$$+ K_{11}\sin\ \frac{6\pi x}{a}\ \sin\ \frac{2\pi y}{a} + K_4\ \sin\frac{4\pi x}{a}\ \sin\frac{2\pi y}{a}$$

$$+ K_{15}\sin\ \frac{4\pi x}{a}\ \sin\ \frac{4\pi y}{a} + 3C_{17}\ \sin\frac{6\pi x}{a}\ \sin\frac{6\pi y}{a}$$

$$+ 3C_{21}\sin\ \frac{4\pi x}{a}\ \sin\ \frac{6\pi y}{a} + 2C_{23}\ \sin\frac{6\pi x}{a}\ \sin\frac{4\pi y}{a} \Big] \tag{5.8}$$

where

$$K_1 = C_4 + C_8 + C_{10} + C_{22}$$

$$K_2 = C_5 + C_9 + C_{12} + C_{14} + C_{18} .$$

$$K_3 = 2C_9 + 2C_{14} + 2C_{18}$$

$$K_4 = C_{13} + C_{15} + C_{19}$$

$$K_6 = 3C_6 + 3C_{16}$$

$$K_7 = 3C_7 + 3C_{20}$$

$$K_8 = 2C_8 + 2C_{22}$$

$$K_{11} = C_{11} + C_{23}$$

$$K_{15} = 2C_{15} + 2C_{19} \tag{5.9}$$

For a square plate, it can be proved that

$$\int_0^{a/2}\int_0^{a/2} \left(\frac{\partial u}{\partial y}\right)^2 dxdy = \int_0^{a/2}\int_0^{a/2} \left(\frac{\partial v}{\partial x}\right)^2 dxdy$$

For $\nu = .316$

$$(1-\nu^2) = 0.9.$$

$$\frac{1-\nu}{2} \int_0^{a/2}\int_0^{a/2} \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] dxdy = .342 \left[\frac{\pi^4}{48} C_2^2 + \frac{\pi^2}{8} C_3^2\right.$$

$$+ \frac{\pi^4}{48} K_1^2 + \frac{\pi^2}{8} K_2^2 + \frac{\pi^4}{48} K_6^2 + \frac{\pi^2}{8} K_7^2 + \frac{\pi^4}{48} K_8^2 + \frac{\pi^2}{8} K_3^2 + \frac{\pi^2}{8} K_{11}^2$$

$$+ \frac{\pi^2}{8} K_4^2 + \frac{\pi^2}{8} K_{15}^2 + \pi C_2 C_3 + \frac{\pi^3}{9} C_2 K_1 + \frac{2\pi}{3} C_2 K_2 + \frac{\pi^3}{35} C_2 K_6$$

$$+ \frac{6\pi}{35} C_2 K_7 - 2\frac{\pi^3}{45} C_2 K_8 - \frac{4\pi}{15} C_2 K_3 + \frac{2\pi}{9} C_2 K_{11} - \frac{\pi}{3} C_2 K_4 + \frac{2\pi}{15} C_2 K_{15}$$

$$+ \frac{8}{3} C_3 K_1 + \frac{16}{9} C_3 K_2 + \frac{24}{35} C_3 K_6 + \frac{16}{35} C_3 K_7 - \frac{16}{15} C_3 K_8 - \frac{32}{45} C_3 K_3$$

$$+ \frac{16}{35} C_3 K_{11} - \frac{32}{45} C_3 K_4 + \frac{64}{225} C_3 K_{15} + \frac{\pi^2}{4} K_1 K_2 + \frac{\pi^2}{12} K_1 K_{11} - \frac{\pi^2}{8} K_1 K_4$$

$$+ \frac{\pi^2}{4} K_6 K_7 + \frac{\pi^2}{4} K_3 K_8 - \frac{\pi^2}{8} K_8 K_{15} + \frac{9}{8} \pi^2 C_{17}^2 + \frac{9}{8} \pi^2 C_{21}^2 + \frac{\pi^2}{2} C_{23}^2$$

$$+ \frac{6\pi}{35} C_2 C_{17} - \frac{9}{35} \pi C_2 C_{21} - \frac{8}{45} \pi C_2 C_{23} + \frac{432}{1225} C_3 C_{17} - \frac{96}{175} C_3 C_{21}$$

$$- \frac{64}{175} C_3 C_{23} + \frac{\pi^2}{6} C_{23} K_8 + \frac{\pi^2}{4} K_6 C_{17} - \frac{3}{8} \pi^2 C_{21} K_6] \qquad (5.10)$$

Similarly

$$\frac{\partial v}{\partial x} = - \frac{\pi}{a} [C_2 \sin\frac{\pi x}{a}(\frac{\pi y}{a}) + C_3 \sin \frac{\pi x}{a} \sin\frac{\pi y}{a} + K_1 \sin \frac{2\pi x}{a}(\frac{\pi y}{a})$$

$$+ K_2 \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{a} + K_6 \sin \frac{6\pi x}{a}(\frac{\pi y}{a}) + K_7 \sin \frac{6\pi x}{a} \sin \frac{2\pi y}{a}$$

$$+ K_8 \sin \frac{4\pi x}{a}(\frac{\pi y}{a}) + K_3 \sin \frac{4\pi x}{a} \sin \frac{2\pi y}{a} + K_{11} \sin \frac{2\pi x}{a} \sin \frac{6\pi y}{a}$$

$$+ K_4 \sin \frac{2\pi x}{a} \sin \frac{4\pi y}{a} + K_{15} \sin \frac{4\pi x}{a} \sin\frac{4\pi y}{a} + 3C_{21} \sin\frac{6\pi x}{a} \sin\frac{4\pi y}{a}$$

$$+ 2C_{23} \sin \frac{4\pi x}{a} \sin \frac{6\pi y}{a} + 3C_{17} \sin \frac{6\pi x}{a} \sin \frac{6\pi y}{a}] \qquad (5.11)$$

where

$K_1, K_2, \ldots, K_{15}$ are defined by (5.9).

$$
\begin{aligned}
\frac{1-\nu}{2} \int_0^{a/2} \int_0^{a/2} 2\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\, dx dy = .342\, [& 2C_2^2 + \frac{\pi^2}{8} C_3^2 + \frac{\pi^2}{8} K_1^2 + \\
& + \frac{\pi^2}{8} K_2^2 + \frac{\pi^2}{72} K_6^2 + \frac{\pi^2}{8} K_{15}^2 + \frac{\pi^2}{32} K_8^2 + \pi C_2 C_3 + \pi C_2 K_1 + \\
& + \frac{2\pi}{3} C_2 K_2 + \frac{\pi}{3} C_2 K_6 + \frac{2}{9} \pi C_2 K_7 - \frac{\pi}{2} C_2 K_8 - \frac{\pi}{3} C_2 K_3 + \frac{6}{35} \pi C_2 K_{11} - \\
& - \frac{4\pi}{15} C_2 K_4 + \frac{2\pi}{15} C_2 K_{15} + \frac{8}{3} K_1 C_3 + \frac{16}{9} C_3 K_2 + \frac{24}{35} C_3 K_6 + \frac{16}{35} C_3 K_7 - \\
& - \frac{16}{15} C_3 K_8 - \frac{32}{45} C_3 K_3 + \frac{16}{35} C_3 K_{11} - \frac{32}{45} C_3 K_4 + \frac{64}{225} C_3 K_{15} + \frac{\pi^2}{4} K_1 K_2 + \\
& + \frac{\pi^2}{12} K_1 K_6 + \frac{\pi^2}{12} K_1 K_7 - \frac{\pi^2}{8} K_1 K_8 - \frac{\pi^2}{8} K_1 K_3 - \frac{\pi^2}{24} K_6 K_8 + \frac{\pi^2}{4} K_6 K_{11} + \\
& + \frac{\pi^2}{4} K_7 K_{11} + \frac{\pi^2}{4} K_4 K_8 - \frac{1}{8} \pi^2 K_8 K_{15} + \frac{\pi^2}{4} K_3 K_4 - \frac{4\pi}{15} C_2 C_{21} - \\
& - \frac{96}{175} C_3 C_{21} + \frac{\pi^2}{4} C_{21} K_8 - \frac{12}{70} \pi C_{23} C_2 - \frac{64}{175} C_{23} C_3 - \frac{\pi^2}{4} C_{23} K_6 + \\
& + \frac{12}{70} \pi C_{17} C_2 + \frac{432}{1225} C_3 C_{17} + \frac{\pi^2}{4} C_{17} K_6 + \frac{3}{2} \pi^2 C_{21} C_{23} + \frac{9}{8} \pi^2 C_{17}^2]
\end{aligned}
\tag{5.12}
$$

From Equations (5.5) and (5.8), we can get

$$\frac{\partial u}{\partial y} \cdot \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial y}$$

For a square plate:

$$\int_0^{a/2}\int_0^{a/2} \left[\frac{\partial u}{\partial y} \cdot \frac{\partial \omega}{\partial x} \cdot \frac{\partial \omega}{\partial y}\right] dxdy = \int_0^{a/2}\int_0^{a/2} \left[\frac{\partial v}{\partial x} \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial y}\right] dxdy$$

Therefore

$$\left(\frac{1-\nu}{2}\right)\int_0^{a/2}\int_0^{a/2} \left(2\frac{\partial u}{\partial y} \cdot \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial y} + 2\frac{\partial v}{\partial x} \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial y}\right) dxdy$$

$$= \frac{.342}{a}\Big[-\frac{\pi^2}{6} C_2D_1 - \frac{4}{9}\pi C_3D_1 - \frac{\pi^3}{16} K_1D_1 - \frac{\pi^3}{16} K_2D_1 + \frac{3\pi^2}{20} C_2D_2$$

$$+ \frac{16}{45}\pi C_3D_2 + \frac{\pi^3}{32} K_1D_2 - \frac{\pi^3}{16} K_8D_2 - \frac{\pi^3}{16} K_3D_2 - \frac{\pi^2}{30} C_2D_3 - \frac{16\pi}{225} C_3D_3$$

$$+ \frac{\pi^3}{32} K_8D_3 - \frac{31}{315}\pi^2 C_2D_4 - \frac{8\pi}{35} C_3D_4 - \frac{\pi^3}{48} K_1D_4 - \frac{\pi^3}{16} K_{11}D_4$$

$$- \frac{\pi^3}{16} K_{15}D_3 - \frac{\pi^3}{16} K_4D_2 - \frac{\pi^3}{16} K_6D_4 - \frac{\pi^3}{16} K_7D_4 + \frac{11}{252}\pi^2 C_2D_5$$

$$+ \frac{16\pi}{175} C_3D_5 - \frac{\pi^3}{48} K_8D_5 + \frac{\pi^3}{32} K_6D_5 - \frac{\pi^2}{70} C_2D_6 - \frac{36\pi}{1225} C_3D_6$$

$$- \frac{\pi^3}{48} K_6D_6 - \frac{\pi^3}{8} C_{23}D_5 - \frac{3\pi^3}{16} C_{21}D_5 - \frac{3}{16}\pi^3 C_{17}D_6\Big]$$

Substituting the values of D's from Equation (5.6) gives:

$$\left(\frac{1-\nu}{2}\right)\int_0^{a/2}\int_0^{a/2}\left(2\frac{\partial u}{\partial y}\quad\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y}\right)dxdy =$$

$$= \frac{.342}{a}\Big[-\frac{\pi^2}{6}C_2\delta_1^2 - \frac{4}{9}\pi C_3\delta_1^2 - \frac{\pi^3}{16}K_1\delta_1^2 - \frac{\pi^3}{16}K_2\delta_1^2$$

$$- \frac{7}{16}\pi^3K_1\delta_2^2 - \frac{5}{8}\pi^3K_2\delta_2^2 - \frac{6}{32}\pi^3K_8\delta_2^2 - \frac{\pi^3}{2}K_3\delta_2^2 - \frac{115}{105}\pi^2C_2\delta_2^2$$

$$- \frac{944}{315}\pi C_3\delta_2^2 - \frac{3}{16}\pi^3K_{11}\delta_2^2 - \frac{10}{16}\pi^3K_{15}\delta_2^2 - \frac{\pi^3}{2}K_4\delta_2^2$$

$$- \frac{3}{16}\pi^3K_6\delta_2^2 - \frac{3}{16}\pi^3K_7\delta_2^2 - \frac{9}{70}\pi^2C_2\delta_3^2 - \frac{324}{1225}\pi C_3\delta_3^2$$

$$- \frac{3}{16}\pi^3K_6\delta_3^2 - \frac{27}{16}\pi^3C_{17}\delta_3^2 - \frac{\pi^2}{15}C_2\delta_1\delta_2 - \frac{16}{45}\pi C_3\delta_1\delta_2$$

$$- \frac{\pi^3}{8}K_1\delta_1\delta_2 - \frac{\pi^3}{4}K_2\delta_1\delta_2 - \frac{\pi^3}{4}K_3\delta_1\delta_2 - \frac{\pi^3}{4}K_4\delta_1\delta_2$$

$$- \frac{\pi^3}{4}K_8\delta_1\delta_2 + \frac{24}{30}\pi^2C_2\delta_1\delta_3 + \frac{56}{25}\pi C_3\delta_1\delta_3 + \frac{6\pi^3}{16}K_1\delta_1\delta_3$$

$$+ \frac{6}{16}\pi^3K_2\delta_1\delta_3 + \frac{6}{32}\pi^3K_8\delta_1\delta_3 - \frac{6}{16}\pi^3K_{15}\delta_1\delta_3 - \frac{\pi^2}{15}C_2\delta_2\delta_3$$

$$- \frac{48}{175}\pi C_3\delta_2\delta_3 - \frac{\pi^3}{8}K_1\delta_2\delta_3 - \frac{3}{8}\pi^3K_{11}\delta_2\delta_3 - \frac{3}{8}\pi^3K_7\delta_2\delta_3$$

$$- \frac{\pi^3}{4}K_8\delta_2\delta_3 - \frac{3}{2}\pi^3C_{23}\delta_2\delta_3 - \frac{9}{4}\pi^3C_{21}\delta_2\delta_3\Big] \tag{5.13}$$

5.3 U_b STRAIN ENERGY DUE TO BENDING

The strain energy due to bending is obtained for a simply supported plate, using Equation (3.34)

$$U = \frac{D}{2} \int_0^{a/2} \int_0^{a/2} \left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)^2 dxdy \qquad (3.29)$$

Using:

$$\omega = \delta_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a}$$

$$+ \delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} + \delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a}$$

gives

$$\left(\frac{\partial^2\omega}{\partial x^2}\right)^2 = \frac{\pi^4}{a^4} \left[\delta_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a} +\right.$$

$$\left. + 9\delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} + 9\delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a}\right]^2$$

For a square plate:

$$\int_0^{a/2} \int_0^{a/2} \left(\frac{\partial^2\omega}{\partial x^2}\right)^2 dxdy = \int_0^{a/2} \int_0^{a/2} \left(\frac{\partial^2\omega}{\partial y^2}\right)^2 dxdy$$

Also making use of the orthogonality property of Fourier series, and using the integration table in Appendix A.

$$\frac{D}{2}\int_0^{a/2}\int_0^{a/2}\left[\left(\frac{\partial^2\omega}{\partial x^2}\right)^2+\left(\frac{\partial^2\omega}{\partial y^2}\right)^2\right]dxdy = \frac{D}{2}\ \frac{\pi^4}{a^2}\left[\frac{\delta_1^2}{8}+\right.$$

$$\left.+\frac{41}{4}\delta_2^2+\frac{81}{8}\delta_3^2\right] \qquad (5.14)$$

Similarly,

$$\frac{D}{2}\int_0^{a/2}\int_0^{a/2} 2\left(\frac{\partial^2\omega}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2}\right)dxdy = \frac{D}{2}\ \frac{\pi^4}{a^2}\left[\frac{\delta_1^2}{8}+\right.$$

$$\left.+\frac{9}{4}\delta_2^2+\frac{81}{8}\delta_3^2\right] \qquad (5.15)$$

Adding Equations (5.14) and (5.15)

$$U_b = \frac{\pi^4}{a^2}D\left[\frac{\delta_1^2}{8}+\frac{100}{16}\delta_2^2+\frac{81}{8}\delta_3^2\right] \qquad (5.16)$$

5.4 U_s STRAIN ENERGY DUE TO COMPRESSION IN THE STIFFENERS

For the considered domain of the problem, i.e., for $0 \leq x \leq \frac{a}{2}$ and $0 \leq y \leq \frac{a}{2}$ the stiffener's strain energy due to axial stress, by virtue of Equation (3.36), is:

$$U_s = \tfrac{1}{2}\int_0^{a/2} \varepsilon_y\,\sigma_y\Big|_{x=\frac{a}{2}} A_s\,dy + \tfrac{1}{2}\int_0^{a/2} \varepsilon_x\,\sigma_x\Big|_{y=\frac{a}{2}} A_s dx$$

$$(5.17)$$

From the symmetry of the plate and loading, and using:

$$\varepsilon_x = \frac{1}{E}\,\sigma_x \tag{5.18}$$

and

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{5.19}$$

gives

$$U_s = A_s\,E \int_0^{a/2} \left(\frac{\partial u}{\partial x}\right)^2 \Bigg|_{y=\frac{a}{2}} dx \tag{5.20}$$

Dividing U_s by $\frac{(a)(Et)}{2(1-\nu^2)}$ and using Equation (5.23) gives:

$$U_s' = U_s\,\frac{2(1-\nu^2)}{Et} = \frac{b}{a}\,(2)(1-\nu^2)\,\frac{\pi^2}{4}\,\frac{C_1^2}{a}$$

After minimization, i.e.,

$$\frac{\partial U_s'}{\partial C_1} = \frac{b}{a}\,(4)(1-\nu^2)\,\frac{\pi^2}{4}\,\frac{C_1}{a} = \frac{b}{a}(1-\nu^2)\pi^2\,C_1'$$

where:

$$C_1' = \frac{C_1}{a}$$

This value will be added to the element $A(1,1)$ in the stiffness matrix $[A]_{28\times 28}$ of the system. Obviously, this value is a function of the stiffener's area, or the ratio $\frac{b}{a}$.

Table (5.1) which follows gives the value of the element $A(1,1)$ of the stiffness matrix $[A]$ of the system, corresponding to five different values of the ratio $\frac{a}{b}$ and for $\nu = .316$

TABLE 5.1

a/b	Non-Stiffened Plate ∞	10	5	2.5	.0947
A(1,1)	6.20	7.09	7.98	9.75	100.0

$$\varepsilon_x^2 = \frac{\pi^2}{a^2}\left[C_1^2 \cos^2\frac{\pi x}{a} + C_2^2 \cos^2\frac{\pi y}{a} + C_3^2 \cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\right.$$

$$+ 2C_1C_3 \cos^2\frac{\pi x}{a}\cos\frac{\pi y}{a} + 2C_1C_2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}$$

$$\left.+ 2C_2C_3 \cos\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\right]$$

$$\varepsilon_x^2\Big|_{y=\frac{a}{2}} = \frac{\pi^2}{a^2} C_1^2 \cos^2\frac{\pi x}{a} \qquad (5.21)$$

$$\therefore \quad U_s = A_s \; E \int_0^{a/2} \frac{\pi^2}{a^2} C_1^2 \cos^2\frac{\pi x}{a}\, dx$$

$$U_s = A_s \; E\left(\frac{\pi^2}{4}\right)\left(\frac{C_1^2}{a}\right) \qquad (5.22)$$

The area of the stiffener is:

$$A_s = b\,t \qquad (5.23)$$

where

t is assumed to be equal to the plate thickness, and b is the stiffener width.

5.5 V, THE POTENTIAL ENERGY OF THE LOAD

The potential energy (V) of the load, referred to the undisturbed plate level as a datum, is clearly

$$\int_0^{a/2}\int_0^{a/2} - \omega q \, dxdy = - q \int_0^{a/2}\int_0^{a/2} \omega \, dxdy \tag{5.24}$$

$$\omega = \delta_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a}$$

$$+ \delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} + \delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a} \tag{4.13}$$

Using integration table in Appendix A

$$\therefore \quad V = - q \left[\frac{a^2}{\pi^2}\delta_1 - \frac{a^2}{3\pi^2}\delta_2 - \frac{a^2}{3\pi^2}\delta_2 + \frac{a^2}{9\pi^2}\delta_3\right]$$

$$V = - \frac{qa}{\pi^2}\left[\delta_1 - \frac{2}{3}\delta_2 + \frac{\delta_3}{9}\right] a \tag{5.25}$$

5.6 THE TOTAL POTENTIAL ENERGY OF THE SYSTEM

$$\pi = U_m + U_b + U_s + V \tag{3.30}$$

Using expressions (5.2), (5.16), (5.22) and (5.25), this becomes:

$$\therefore \quad \pi = \frac{Et}{2(1-\nu^2)} U_m(C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11},$$

$$K_{15}, C_{17}, C_{21}, C_{23}, \delta_1, \delta_2, \delta_3) + \frac{Et^3}{12(1-\nu^2)a^2} U_b(\delta_1, \delta_2, \delta_3)$$

$$+ A_s \cdot \frac{E}{a} U_s(C_1) + qa^2 \; V(\delta_1, \delta_2, \delta_3) \qquad (5.26)$$

$$\pi = k \; U_m(C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11}, K_{15}, C_{17},$$

$$C_{21}, C_{23}, \delta_1, \delta_2, \delta_3) + \frac{D}{a^2} U_b \; (\delta_1, \delta_2, \delta_3)$$

$$+ A_s \cdot \frac{E}{a} U_s(C_1) + qa^2 \; V(\delta_1, \delta_2, \delta_3) \qquad (5.27a)$$

where

$$k = \frac{Et}{2(1-\nu^2)} \quad \text{and} \quad D = \frac{Et^3}{12(1-\nu^2)} \qquad (5.27b)$$

$$\therefore \quad \pi = k\{U_m(C_1, C_2, \ldots, K_1, \ldots, C_{23}, \delta_1, \delta_2, \delta_3)$$

$$+ \frac{D}{a^2 k} U_b(\delta_1, \delta_2, \delta_3) + \frac{A_s \; E}{ka} \; U_s \; (C_1)$$

$$+ (\frac{qa}{k}) a.V(\delta_1, \delta_2, \delta_3)\} \qquad (5.28)$$

where

$$\frac{q \cdot a}{k} = \frac{q}{E}\left(\frac{a}{t}\right)2(1-\nu^2) \qquad \frac{D}{a^2 k} = \frac{1}{6}\left(\frac{t}{a}\right)^2$$

5.7 CASE OF NON-STIFFENED PLATES

In the case of non-stiffened plates, $A_s = 0$, the term of U_s vanishes and expression becomes

$$\pi = k\{U_m(C_1, C_2, \ldots, K_1, \ldots, C_{23}, \delta_1, \delta_2, \delta_3)$$

$$+ \frac{D}{a^2 k} U_b(\delta_1, \delta_2, \delta_3) + \left(\frac{qa}{k}\right) aV(\delta_1, \delta_2, \delta_3)\} \qquad (5.29)$$

Substituting expressions (5.2), (5.16) and (5.25), in equation (5.29) and for $(\frac{a}{t}) = 200$, $\nu = .316$.

$$\therefore \quad \pi = k[3.0994C_1^2 + 4.47744C_2^2 + 2.46735C_3^2 + 1.11597\,K_1^2$$

$$+ .84385K_2^2 + .4219K_3^2 + .4219K_4^2 + .74092K_6^2 + .4219K_7^2$$

$$+ .79954K_8^2 + .4219K_{11}^2 + .8438K_{15}^2 + 7.5946C_{17}^2 + 3.7973C_{21}^2$$

$$+ 1.6877C_{23}^2 + 5.5594C_1C_2 + 4.1343C_1C_3 + 6.2831C\,C_2C_3$$

$$+ 2.2526C_2K_1 + 1.4326C_2K_2 - .6446\,C_2K_3 - .6446C_2K_4$$

$$+ .66108C_2K_6 + .42296C_2K_7 - 1.0085C_2K_8 + .423C_2K_{11}$$

$$+ .2865C_2K_{15} + .3684C_2C_{17} - .5628C_2C_{21} - .3752C_2C_{23}$$

$$+ 1.824C_3K_1 + 1.216C_3K_2 - .4864C_3K_3 - .4864C_3K_4$$

$$+ .469C_3K_6 + .31268C_3K_7 - .7296C_3K_8 + .31268C_3K_{11}$$

$$+ .19456C_3K_{15} + .24122C_3C_{17} - .3752C_3C_{21} - .2502C_3C_{23}$$

$$+ 1.6877K_1K_2 - .4219K_1K_3 - .4219K_1K_4 + .2813K_1K_6$$

$$+ .2813K_1K_7 - .4219K_1K_8 + .84385K_3K_4 + .84385K_3K_8 + .84385K_4K_8$$

$$+ .84385K_6K_7 - .1406K_6K_8 + .84385K_6K_{11} + 1.6877K_6C_{17}$$

$$- .84385K_6C_{23} - 1.2658K_6C_{21} + .84385K_7K_{11} - .8438K_8K_{15}$$

$$+ .56257K_8C_{23} + .84385K_8C_{21} + 5.0631C_{21}C_{23}$$

$$\frac{1}{a}(-.5626C_2\delta_1^2 - 3.6969C_2\delta_2^2 - .434C_2\delta_3^2 - .225C_2\delta_1\delta_2$$

$$+ 2.7003C_2\delta_1\delta_3 - .225C_2\delta_2\delta_3 - .4775C_3\delta_1^2 - 3.2199C_3\delta_2^2$$

$$+ .2842C_3\delta_3^2 - .382C_3\delta_1\delta_2 + 2.4067C_3\delta_1\delta_3 - .2947C_3\delta_2\delta_3$$

$$- .6628K_1\delta_1^2 - 4.6393K_1\delta_2^2 - 1.3255K_1\delta_1\delta_2 + 3.9766K_1\delta_1\delta_3$$

$$- 1.3255K_1\delta_2\delta_3 - .6628K_2\delta_1^2 - 6.6276K_2\delta_2^2 - 2.651K_2\delta_1\delta_2$$

$$+ 3.9766K_2\delta_1\delta_3 - 5.3021K_3\delta_2^2 - 2.651K_3\delta_1\delta_2 - 5.3021K_4\delta_2^2$$

$$- 2.651K_4\delta_1\delta_2 - 1.9883K_6\delta_2^2 - 1.9883K_6\delta_3^2 - 1.9883K_7\delta_2^2$$

$$- 3.9766K_7\delta_2\delta_3 - 1.9883K_8\delta_2^2 - 2.651K_8\delta_1\delta_2 + 1.9883K_8\delta_1\delta_3$$

$$- 2.651K_8\delta_2\delta_3 - 1.9883K_{11}\delta_2^2 - 3.9766K_{11}\delta_2\delta_3 - 6.6276K_{15}\delta_2^2$$

$$- 3.9766K_{15}\delta_1\delta_3 - 17.8945C_{17}\delta_3^2 - 15.9062C_{23}\delta_2\delta_3$$

$$- 23.8593C_{21}\delta_2\delta_3) + \frac{1}{a^2}(.13013\delta_1^4 + 1.04106\delta_1^3\delta_2$$

$$+ 8.849\delta_1^2\delta_2^2 + 27.0676\delta_1\delta_2^3 - 1.5616\delta_1^3\delta_3 + 9.3695\delta_1^2\delta_3^2$$

$$- 6.24636\delta_1^2\delta_2\delta_3 + 45.0258\delta_2^4 + 46.8477\delta_2^2\delta_3^2 + 9.3695\delta_2\delta_3^3$$

$$+ 10.5407\delta_3^4) + 5.0734 \times 10^{-5}\delta_1^2 - 9.1204 \times 10^{-6}\delta_1(a)$$

$$+ 2.5367 \times 10^{-3}\delta_2^2 + 6.0802 \times 10^{-6}\delta_2(a) + 4.1094 \times 10^{-3}\delta_3^2$$

$$- 1.0134 \times 10^{-6}\delta_3(a)] \qquad (5.30)$$

5.8 LAGRANGIAN MULTIPLIER TECHNIQUE

It is to be noted that not all the variables of U_m are independent. Expression (4.17) implies:

$$C_2 = -\nu C_1 \qquad (5.31)$$

Also, expression (5.9) and from expression (4.17)

$$K_1 = C_4 + C_8 + C_{10} + C_{22} = \frac{1}{a}(-.7854\delta_1^2 - 1.5708\delta_1\delta_2 - 7.0686\delta_2^2$$

$$- 14.1372\delta_2\delta_3)$$

$$K_2 = C_5 + C_9 + C_{12} + C_{14} + C_{18} = \frac{1}{a}(.3927\delta_1^2 + .7854\delta_1\delta_2$$

$$- 2.3562\delta_1\delta_2 - 2.3562\delta_2^2 - 2.3562\delta_1\delta_3)$$

$$K_3 = 2C_9 + 2C_{14} + 2C_{18} = (1.5708\delta_1\delta_2 - 4.7124\delta_2^2 - 4.7124\delta_1\delta_3)\frac{1}{a}$$

$$K_4 = C_{13} + C_{15} + C_{19} = (1.1781\delta_1\delta_2 + 1.1781\delta_2^2 + 1.1781\delta_1\delta_3)\frac{1}{a}$$

$$K_6 = 3C_6 + 3C_{16} = (-2.3562\delta_2^2 - 21.2058\delta_3^2)\frac{1}{a}$$

$$K_7 = 3C_7 + 3C_{20} = (1.1781\delta_2^2 - 7.0686\delta_2\delta_3)\frac{1}{a}$$

$$K_8 = 2C_8 + 2C_{22} = (-3.1416\delta_1\delta_2 - 28.2743\delta_2\delta_3)\frac{1}{a}$$

$$K_{11} = C_{11} + C_{23} = (1.1781\delta_2^2 + 2.3562\delta_2\delta_3)\frac{1}{a}$$

$$K_{15} = 2C_{15} + 2C_{19} = (2.3562\delta_2^2 + 2.3562\delta_1\delta_3)\frac{1}{a}\,. \qquad (5.31)$$

while expression (4.17)

$$C_{17} = 1.1781\,\frac{\delta_3^2}{a}$$

$$C_{21} = 1.1781\,\frac{\delta_2\delta_3}{a}$$

$$C_{23} = 2.3562\,\frac{\delta_2\delta_3}{a} \qquad (5.31)$$

Relations (5.31) can be more conveniently expressed in the form:

$$g_1(C_1,C_2) = (.316C_1+C_2) = 0$$

$$g_2(K_1,\delta_1,\delta_2,\delta_3) = (K_1+.7854\frac{\delta_1^2}{a}+1.5708\frac{\delta_1\delta_2}{a}+7.0686\frac{\delta_2^2}{a} + 14.1372\frac{\delta_2\delta_3}{a}) = 0$$

$$g_3(K_2,\delta_1,\delta_2,\delta_3) = (K_2-.3927\frac{\delta_1^2}{a}+1.5708\frac{\delta_1\delta_2}{a}+2.3562\frac{\delta_2^2}{a} + 2.3562\frac{\delta_1\delta_3}{a} = 0$$

$$g_4(K_3,\delta_1,\delta_2,\delta_3) = (K_3-1.5708\frac{\delta_1\delta_2}{a}+4.7124\frac{\delta_2^2}{a} + 4.7124\frac{\delta_1\delta_3}{a}) = 0$$

$$g_5(K_4,\delta_1,\delta_2,\delta_3) = (K_4-1.1781\frac{\delta_1\delta_2}{a}-1.1781\frac{\delta_2^2}{a} - 1.1781\frac{\delta_1\delta_3}{a}) = 0$$

$$g_6(K_6,\delta_2,\delta_3) = (K_6+2.3562\frac{\delta_2^2}{a}+21.2058\frac{\delta_3^2}{a}) = 0$$

$$g_7(K_7,\delta_2,\delta_3) = (K_7-1.1781\frac{\delta_2^2}{a}+7.0686\frac{\delta_2\delta_3}{a}) = 0$$

$$g_8(K_8,\delta_1,\delta_2,\delta_3) = (K_8+3.1416\frac{\delta_1\delta_2}{a}+28.2743\frac{\delta_2\delta_3}{a}) = 0$$

$$g_9(K_{11},\delta_2,\delta_3) = (K_{11}-1.1781\frac{\delta_2^2}{a}-2.3562\frac{\delta_2\delta_3}{a}) = 0$$

$$g_{10}(K_{15},\delta_1,\delta_2,\delta_3) = (K_{15}-2.3562\frac{\delta_2^2}{a}-2.3562\frac{\delta_1\delta_3}{a}) = 0$$

$$g_{11}(C_{17},\delta_3) = (C_{17}-1.1781\frac{\delta_3^2}{a}) = 0$$

$$g_{12}(C_{21},\delta_2,\delta_3) = (C_{21}-1.1781\frac{\delta_2\delta_3}{a}) = 0$$

$$g_{13}(C_{23},\delta_2,\delta_3) = (C_{23}-2.3562\frac{\delta_2\delta_3}{a}) = 0 \qquad (5.32)$$

Therefore, the problem of minimization of the function π subjected to certain specified constraint conditions is encountered.

We wish to minimize the function

$$\pi(C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11}, K_{15}, C_{17}, C_{21}, C_{23}, \delta_1, \delta_2, \delta_3),$$

where π is a function of 18 variables, subjected to the constraints $g_i' = 0$, where $i = 1,2,\ldots, 13$.

The problem is handled by the Lagrangian multiplier method. Instead of minimizing the function π directly, we construct a new function $\overline{\pi}$ where:

$$\overline{\pi}(C_1, \ldots, \delta_3, \lambda_i) = \pi(C_1, \ldots, \delta_3) + \sum_{i=1}^{13} \lambda_i g_i \qquad (5.33)$$

i.e., the case of n ($n = 18$) independent variables ($C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11}, K_{15}, C_{17}, C_{21}, C_{23}, \delta_1, \delta_2, \delta_3$) and m ($m = 13$) constraint conditions (where $m < n$), given by expressions (5.32) $g_i = 0$.

The local minimum of π can be found by solving the $n + m$) equations

$$\left.\begin{aligned} \frac{\partial \overline{\pi}}{\partial C_i} &= 0 \qquad i=1,2,\ \ldots,\ n \\ \frac{\partial \overline{\pi}}{\partial \lambda_j} &= 0 \qquad j=1,2,\ \ldots,\ m \end{aligned}\right\} \qquad (5.34)$$

If a local minimum of $\overline{\pi}$ occurs at

$$(C_1^*, C_2^*, \ldots, \delta_3^*, \lambda_1^*, \ldots, \lambda_{13}^*),$$

then $\pi(C_1^*, C_2^*, \ldots, \delta_3^*)$ is a local minimum of π subject to $g_i = 0$.

5.9 MINIMIZATION PROCEDURE AND SOLUTION

Pre-multiplying each equation of expression (5.32) by λ_i (where $i=1,2, \ldots, 13$) and then adding the whole expression to π, we construct the function $\overline{\pi}$ as given by equation (5.33). Partial differentiation of the new function $\overline{\pi}$ with respect to the variables $C_1, C_2, \ldots, K_1, \ldots, K_{15}, C_{23}, \lambda_1, \lambda_2, \ldots, \lambda_{13}$, and equating each equation to zero, we have a system of 28 equations to be solved, given below.

$$\frac{\partial \overline{\pi}}{\partial C_1} = 0$$

$$6.1988C_1 + 5.5594C_2 + 4.1343C_3 + .316\lambda_1 = 0$$

$$\frac{\partial \overline{\pi}}{\partial C_2} = 0$$

$$5.5594C_1 + 8.9549C_2 + 6.2831C_3 + 2.2526K_1 + 1.4326K_2$$

$$- .6446K_3 - .6446K_4 + .66108K_6 + .42296K_7 - 1.0085K_8$$

$$+ .423K_{[illegible]} + .28656K_{15} + .3684C_{17} - .5628C_{21} - .3752C_{23}$$

$$+ \lambda_1 = \frac{1}{a}(.5626\delta_1^2 + 3.6969\delta_2^2 + .434\delta_3^2 + .225\delta_1\delta_2 - 2.7003\delta_1\delta_3 + .225\delta_2\delta_3)$$

$$\frac{\partial\bar{\pi}}{\partial C_3} = 0$$

$$4.1343C_1 + 6.2831C_2 + 4.9347C_3 + 1.824K_1 + 1.216K_2$$

$$- .4864K_3 - .4864K_4 + .469K_6 + .31268K_7 - .7296K_8$$

$$- .31268K_{11} + .19456K_{15} - .24122C_{17} - .3752C_{21} - .2502C_{23} -$$

$$= \frac{1}{a}(.4775\delta_1^2+3.2199\delta_2^2+.2842\delta_3^2+.382\delta_1\delta_2-2.4067\delta_1\delta_3+.2947\delta_2\delta_3)$$

$$\frac{\partial\bar{\pi}}{\partial K_1} = 0$$

$$2.2526C_2 + 1.824C_3 + 2.23194K_1 + 1.6877K_2 - .4219K_3$$

$$- .4219K_4 + .2813K_6 + .2813K_7 - .4219K_8 + .2813K_{11}$$

$$+ \lambda_2 = \frac{1}{a}(.6628\delta_1^2+4.6393\delta_2^2+1.3255\delta_1\delta_2-3.9766\delta_1\delta_3+1.3255\delta_2\delta_3)$$

$$\frac{\partial\bar{\pi}}{\partial K_2} = 0$$

$$1.4326C_2 + 1.216C_3 + 1.6877K_1 + 1.6877K_2 + \lambda_3 =$$

$$\frac{1}{a}(.6628\delta_1^2+6.6276\delta_2^2+2.651\delta_1\delta_2-3.9766\delta_1\delta_3)$$

$$\frac{\partial\bar{\pi}}{\partial K_3} = 0$$

$$- .6446C_2 - .4864C_3 - .4219K_1 + .84385K_3 + .84385K_4$$

$$+ .84385K_8 + \lambda_4 = (5.3021\delta_2^2+2.651\delta_1\delta_2)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial K_4} = 0$$

$$- .6446C_2 - .4864C_3 - .4219K_1 + .84385K_3 + .84385K_4$$

$$+ .84385K_8 + \lambda_5 = (5.3021\delta_2^2+2.651\delta_1\delta_2)\frac{1}{a}$$

$$\frac{\partial\overline{\pi}}{\partial K_6} = 0$$

$$.66108C_2 + .469C_3 + .2813K_1 + 1.48184K_6 + .84385K_7$$

$$- .1406K_8 + .84385K_{11} + 1.6877C_{17} - 1.2658C_{21}$$

$$- .84385C_{23} + \lambda_6 = (1.9883\delta_2^2+1.9883\delta_3^2)\frac{1}{a}$$

$$\frac{\partial\overline{\pi}}{\partial K_7} = 0$$

$$.42296C_2 + .31268C_3 + .2813K_1 + .84385K_6 + .84385K_7$$

$$+ .84385K_{11} + \lambda_7 = (1.9883\delta_2^2+3.9766\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\overline{\pi}}{\partial K_8} = 0$$

$$- 1.0085C_2 - .7296C_3 - .4219K_1 + .84385K_3 + .84385K_4$$

$$- .1406K_6 + 1.5991K_8 - .8438K_{15} + .84385C_{21} + .56257C_{23}$$

$$+ \lambda_8 = (1.9883\delta_2^2+2.651\delta_1\delta_2-1.9883\delta_1\delta_3+2.651\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\overline{\pi}}{\partial K_{11}} = 0$$

$$.423C_2 + .31268C_3 + .2813K_1 + .84385K_6 + .84385K_7$$

$$+ .84385K_{11} + \lambda_9 = (1.9883\delta_2^2+3.9766\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\overline{\pi}}{\partial K_{15}} = 0$$

$$.28656C_2 + .19456C_3 - .8438K_8 + 1.6876K_{15} + \lambda_{10} =$$

$$= (6.6276\delta_2^2 + 3.9766\delta_1\delta_3)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial C_{17}} = 0$$

$$.3684C_2 + .24122C_3 + 1.6877K_6 + 15.1892C_{17} + \lambda_{11}$$

$$= (17.8945\delta_3^2)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial C_{21}} = 0$$

$$-.5628C_2 - .3752C_3 - 1.2658K_6 + .84385K_8 + 7.5946C_{21}$$

$$+ 5.0631C_{23} + \lambda_{12} = (23.8593\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial C_{23}} = 0$$

$$-.3752C_2 - .2502C_3 - .84385K_6 + .56257K_8 + 5.0631C_{21}$$

$$+ 3.3754C_{23} + \lambda_{13} = (15.9062\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial\lambda_1} = 0$$

$$.316C_1 + C_2 = 0$$

$$\frac{\partial\pi}{\partial\lambda_2} = 0$$

$$K_1 = (-.7854\delta_1^2 - 7.0686\delta_2^2 - 1.5708\delta_1\delta_2 - 14.1372\delta_2\delta_3)\frac{1}{a}$$

$$\frac{\partial\bar{\pi}}{\partial\lambda_3} = 0$$

$$K_2 = (.3927\delta_1^2 - 2.3562\delta_2^2 - 1.5708\delta_1\delta_2 - 2.3562\delta_1\delta_3)\frac{1}{a}$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_4} = 0$$

$$K_3 = (-4.7124\delta_2^2 + 1.5708\delta_1\delta_2 - 4.7124\delta_1\delta_3)\frac{1}{a}$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_5} = 0$$

$$K_4 = 1.1781\delta_2^2 + 1.1781\delta_1\delta_2 + 1.1781\delta_1\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_6} = 0$$

$$K_6 = -2.3562\delta_2^2 - 21.2058\delta_3^2$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_7} = 0$$

$$K_7 = 1.1781\delta_2^2 - 7.0686\delta_2\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_8} = 0$$

$$K_8 = -3.1416\delta_1\delta_2 - 28.2743\delta_2\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_9} = 0$$

$$K_{11} = 1.1781\delta_2^2 + 2.3562\delta_2\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_{10}} = 0$$

$$K_{15} = 2.3562\delta_2^2 + 2.3562\delta_1\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_{11}} = 0$$

$$C_{17} = 1.1781\delta_3^2$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_{12}} = 0$$

$$C_{21} = 1.1781\delta_2\delta_3$$

$$\frac{\partial \bar{\pi}}{\partial \lambda_{13}} = 0$$

$$\therefore \quad C_{23} = 2.3562\delta_2\delta_3 \qquad (5.35)$$

These equations are not linear, as in the case of small deflections. Equations (5.35) are linear in the parameters $C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11}, K_{15}, C_{17}, C_{21}, C_{23}$, $\lambda_1, \lambda_2, \ldots, \lambda_{13}$, and quadratic in the parameters $\delta_1, \delta_2, \delta_3$. A solution is obtained by solving Equations (5.35) $C_1, C_2, \ldots, \ldots, \lambda_{13}$ in terms of the $\delta_1, \delta_2, \delta_3$.

The system of equations (5.35) can be put into the form:

$$[A]_{28\times 28}\ [C_1, C_2, C_3, K_1, K_2, K_3, K_4, K_6, K_7, K_8, K_{11}, K_{15}, C_{17}, C_{21}, C_{23}, \lambda_1, \lambda_2, \ldots, \lambda_{13}]^T$$

$$= [B]_{28\times 6}\ [\delta_1^2\ \delta_2^2\ \delta_3^2\ \delta_1\delta_2\ \delta_1\delta_3\ \delta_2\delta_3]^T \qquad (5.36)$$

where

[A] is a square symmetric matrix of size 28×28.

[B] is a rectangular matrix of size 28×6.

Premultiplying both sides of system (5.36) by $[A]^{-1}$.

$$\therefore [C_1, C_2, \ldots, \lambda_{13}]^T_{28\times1} = [A]^{-1}_{28\times28}\,[B]_{28\times6}\,[\delta_1^2, \ldots, \delta_2\delta_3]^T_{6\times1}$$

$$\therefore [C_1, C_2, \ldots, \lambda_{13}]^T_{28\times1} = [C]_{28\times6}\,[\delta_1^2\;\delta_2^2\;\delta_3^2\;\delta_1\delta_2\;\delta_1\delta_3\;\delta_2\delta_3]^T_{6\times1} \tag{5.37}$$

where

$$[C]_{28\times6} = [A]^{-1}_{28\times28}\;[B]_{28\times6}$$

Partial differentiation of the function $\bar{\pi}$ with respect to variables δ_1, δ_2 and δ_3, and equating each equation to zero, we have a system of three equations to be solved, given below:

$$\frac{\partial\bar{\pi}}{\partial\delta_1} = 0$$

$$- 1.1252\,C_2\delta_1 - .955\,C_3\delta_1 - 1.3256\,K_1\delta_1 - 1.3256K_2\delta_1$$

$$+ 1.5708\,\lambda_2\delta_1 - .7854\,\lambda_3\delta_1$$

$$- .225\,C_2\delta_2 - .382C_3\delta_2 - 1.3255\,K_1\delta_2 - 2.651K_2\delta_2 - 2.651K_3\delta_2$$

$$- 2.651K_4\delta_2 - 2.651K_8\delta_2 + 1.5708\lambda_2\delta_2 + 1.5708\lambda_3\delta_2$$

$$- 1.5708\lambda_4\delta_2 - 1.1781\lambda_5\delta_2 + 3.1416\lambda_8\delta_2$$

$$+ 2.7003C_2\delta_3 + 2.4067C_3\delta_3 + 3.9766K_1\delta_3 + 3.9766K_2\delta_3$$

$$+ 1.9883K_8\delta_3 - 3.9766K_{15}\delta_3 + 2.3562\lambda_3\delta_3 + 4.7124\lambda_4\delta_3$$

$$- 1.1781\lambda_5\delta_3 - 2.3562\lambda_{10}\delta_3 + .52052\delta_1^3 + 3.12318\delta_1^2\delta_2$$

$$+ 17.698\delta_1\delta_2^2 + 27.0676\delta_2^3 - 4.6848\delta_1^2\delta_3 + 18.739\delta_1\delta_3^2$$

$$- 12.49272\delta_1\delta_2\delta_3 + 1.01468\times10^{-4}\delta_1 + 9.1204\times10^{-6}\times AK = 0 \qquad (5.38)$$

Equation (5.38) can be put into the form

$$\delta_1[D_{11}]_{1\times28}\,\{C_1C_2\,\ldots\,\lambda_{13}\}^T_{28\times1}$$

$$+\ \delta_2[D_{12}]_{1\times28}\,\{C_1C_2\,\ldots\,\lambda_{13}\}^T$$

$$+\ \delta_3[D_{13}]_{1\times28}\,\{C_1C_2\,\ldots\,\lambda_{13}\}^T$$

$$+ .52052\delta_1^3 + 3.12318\delta_1^2\delta_2 + 17.698\delta_1\delta_2^2 + 27.0676\delta_2^3 - 4.6848\delta_1^2\delta_3$$

$$+ 18.739\delta_1\delta_3^2 - 12.49272\delta_1\delta_2\delta_3 + 1.01468\times10^{-4}\delta_1 - 9.1204\ 10^{-6}$$

$$\times AK = 0 \qquad (5.39)$$

where:

$$\frac{q}{E}\cdot\left(\frac{a}{t}\right)^4 = 400,\ \frac{a}{t} = 200$$

and $AK \leq 1$ is the load factor

$AK = 1$ represents the maximum load

$$\frac{\partial \bar{\pi}}{\partial \delta_2} = 0$$

$$
\begin{aligned}
&- 7.3938C_2\delta_2 - 6.4398C_3\delta_2 - 9.2786K_1\delta_2 - 13.2552K_2\delta_2 \\
&- 10.6042K_3\delta_2 - 10.6042K_4\delta_2 - 3.9766K_6\delta_2 - 3.9766K_7\delta_2 \\
&- 3.9766K_8\delta_2 - 3.9766K_{11}\delta_2 - 13.2552K_{15}\delta_2 + 14.1372\lambda_2\delta_2 \\
&+ 4.7124\lambda_3\delta_2 + 9.4248\lambda_4\delta_2 - 2.3562\lambda_5\delta_2 + 4.7124\lambda_6\delta_2 \\
&- 2.3562\lambda_7\delta_2 - 2.3562\lambda_9\delta_2 - 4.7124\lambda_{10}\delta_2 - .225C_2\delta_1 \\
&- .382C_3\delta_1 - 1.3255K_1\delta_1 - 2.651K_2\delta_1 - 2.651K_3\delta_1 - 2.651K_4\delta_1 \\
&- 2.651K_8\delta_1 + 1.5708\lambda_2\delta_1 + 1.5708\lambda_3\delta_1 - 1.5708\lambda_4\delta_1 \\
&- 1.1781\lambda_5\delta_1 + 3.1416\lambda_8\delta_1 - .225C_2\delta_3 - .2947C_3\delta_3 - 1.3255K_1\delta_3 \\
&- 3.9766K_7\delta_3 - 2.651K_8\delta_3 - 3.9786K_{11}\delta_3 - 23.8593C_{21}\delta_3 \\
&- 15.9062C_{23}\delta_3 + 14.1372\lambda_2\delta_3 + 7.0686\lambda_7\delta_3 + 28.2743\lambda_8\delta_3 \\
&- 2.3562\lambda_9\delta_3 - 1.1781\lambda_{12}\delta_3 - 2.3562\lambda_{13}\delta_3 + 1.04106\delta_1^3 \\
&+ 17.698\delta_1^2\delta_2 + 81.2028\delta_1\delta_2^2 - 6.24636\delta_1^2\delta_3 + 180.1032\delta_2^3 \\
&+ 93.6954\delta_2\delta_3^2 + 28.1085\delta_2^2\delta_3 + 5.0734\times10^{-3}\delta_2 + 6.0802\times10^{-6} \\
&\times AK = 0
\end{aligned}
\tag{5.40}
$$

Equation (5.40) can be put into the form

$$\delta_1[D_{21}]\ \{C_1C_2\ \dots\ \lambda_{13}\}^T$$

$$\delta_2[D_{22}]\ \{C_1C_2\ \dots\ \lambda_{13}\}^T$$

$$\delta_3[D_{23}]\ \{C_1C_2\ \dots\ \lambda_{13}\}^T$$

$$+ 1.041066\delta_1^3 + 17.698\delta_1^2\delta_2 + 81.2028\delta_1\delta_2^2 - 6.246366\delta_1^2\delta_3$$

$$+ 180.1032\delta_2^3 + 93.6954\delta_2\delta_3^2 + 28.1085\delta_2^2\delta_3 + 5.0734\times10^{-3}\delta_2$$

$$+ 6.0802\times10^{-6}\times AK = 0 \qquad (5.41)$$

where:

$$\frac{q}{E}\left(\frac{a}{t}\right)^4 = 400 \qquad \left(\frac{a}{t}\right) = 200$$

$AK \leq 1$ is the load factor

$AK = 1$ represents the maximum load

i.e., at $\frac{q}{E}\left(\frac{a}{t}\right)^4 = 400$

$$\frac{\partial \bar{\pi}}{\partial \delta_3} = 0$$

$$- .868C_2\delta_3 - .5684C_3\delta_3 - 3.9766K_6\delta_3 - 35.789C_{17}\delta_3$$

$$+ 42.4116\lambda_6\delta_3 - 2.3562\lambda_{11}\delta_3 + 2.7003C_2\delta_1 + 2.4067C_3\delta_1$$

$$+ 3.9766K_1\delta_1 + 3.9766K_2\delta_1 + 1.9883K_8\delta_1 - 3.9766K_{15}\delta_1$$

$$+ 2.3562\lambda_3\delta_1 + 4.7124\lambda_4\delta_1 - 1.1781\lambda_5\delta_1 - 2.3562\lambda_{10}\delta_1$$

$$- .225C_2\delta_2 - .2947C_3\delta_2 - 1.3255K_1\delta_2 - 3.9766K_7\delta_2$$

$$- 2.651K_8\delta_2 - 3.9766K_{11}\delta_2 - 15.9062C_{23}\delta_2 - 23.8593C_{21}\delta_2$$

$$+ 14.1372\lambda_2\delta_2 + 7.0686\lambda_7\delta_2 + 28.2743\lambda_8\delta_2 - 2.3562\lambda_9\delta_2$$
$$- 1.1781\lambda_{12}\delta_2 - 2.3562\lambda_{13}\delta_2 - 1.5616\delta_1^3 + 18.739\delta_1^2\delta_3$$
$$- 6.246636\delta_1^2\delta_2 + 93.6954\delta_2^2\delta_3 + 9.3695\delta_2^3 + 42.1628\delta_3^3$$
$$+ 8.2188\times10^{-3}\delta_3 - 1.0134\times10^{-6} \times AK = 0 \tag{5.42}$$

Equation (5.42) can be put into the form

$$\delta_1[D_{31}]\{C_1C_2 \ldots \lambda_{13}\}^T + \delta_2[D_{32}]\{C_1C_2 \ldots \lambda_{13}\}^T$$
$$\delta_3[D_{33}]\{C_1C_2 \ldots \lambda_{13}\}^T$$
$$- 1.5616\delta_1^3 + 18.739\delta_1^2\delta_3 - 6.246636\delta_1^2\delta_2 + 93.6954\delta_2^2\delta_3$$
$$+ 9.3695\delta_2^3 + 42.1628\delta_3^3 + 8.2188\times10^{-3}\delta_3 - 1.0134\times10^{-6}$$
$$\times AK = 0 \tag{5.43}$$

where:

$$\frac{q}{E}\left(\frac{a}{t}\right)^4 = 400 \qquad \left(\frac{a}{t}\right) = 200$$

$AK \le 1$ is the load factor

$AK = 1$ represents the maximum load

i.e., at $\quad \frac{q}{E}\left(\frac{a}{t}\right)^4 = 400$.

It is to be noted that

$$[D_{12}] = [D_{21}]$$

$$[D_{13}] = [D_{31}]$$

and

$$[D_{23}] = [D_{32}] .$$

After solving the system of equations (5.37) for $\{C_1, C_2 \ldots \lambda_{13}\}^T$ in terms of $\delta_1, \delta_2, \delta_3$, the value of $\{C_1, C_2 \ldots \lambda_{13}\}^T$ is substituted in equations (5.39), (5.41), and in (5.43). In this way, we obtain three equations of the third degree involving the parameters δ_1, δ_2, and δ_3, alone. These equations can then be solved numerically, in each particular case of loading by successive approximations.

Numerical values of all the parameters have been computed for ten intensities of the load q, and for five cases of different stiffener areas.

CHAPTER VI

RESULTS

CHAPTER VI

RESULTS

6.1 IN-PLANE DISPLACEMENTS

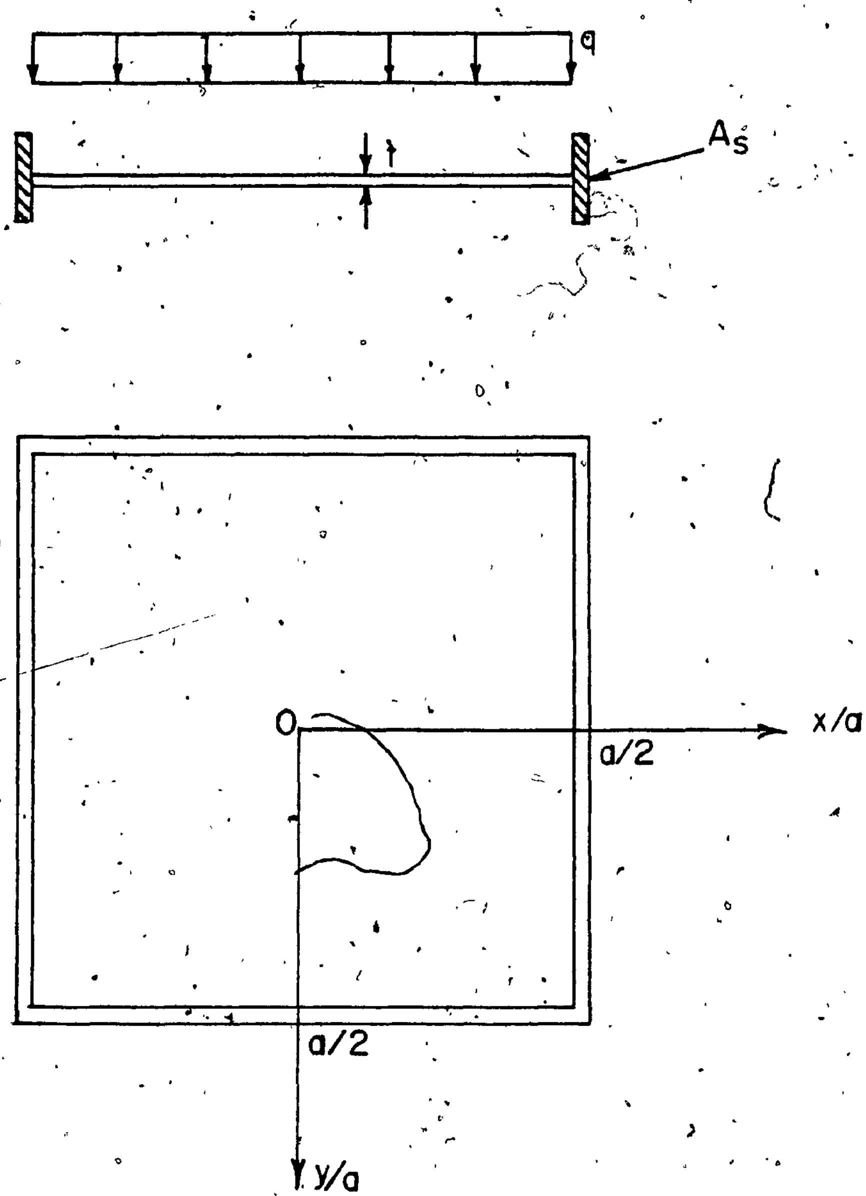
The in-plane displacements u and v are given by

$$u = u(x,y) \qquad (4.15)$$

$$v = v(x,y) \qquad (4.16)$$

The value of the coefficients $C_1, C_2, \ldots, C_{23}$ are calculated for ten different intensities of the load, and for five values of the stiffener area.

The values of the displacements u_1, and u_2, shown in Fig. (6.2) were calculated in each case. The ratio u_1/u_2 under the maximum load level (Q = 400), was compared for the five cases of different stiffener areas. As the stiffener area increases, this ratio decreases and ultimately approaches zero value as the stiffener area tends to infinity. This indicates that a plate with a large stiffener area can be represented as a plate with its corners restrained against in-plane motion.



SQUARE ISOTROPIC PLATE STIFFENED ALONG THE BOUNDARY PLATE BEFORE DEFORMATION

FIG (6.1)

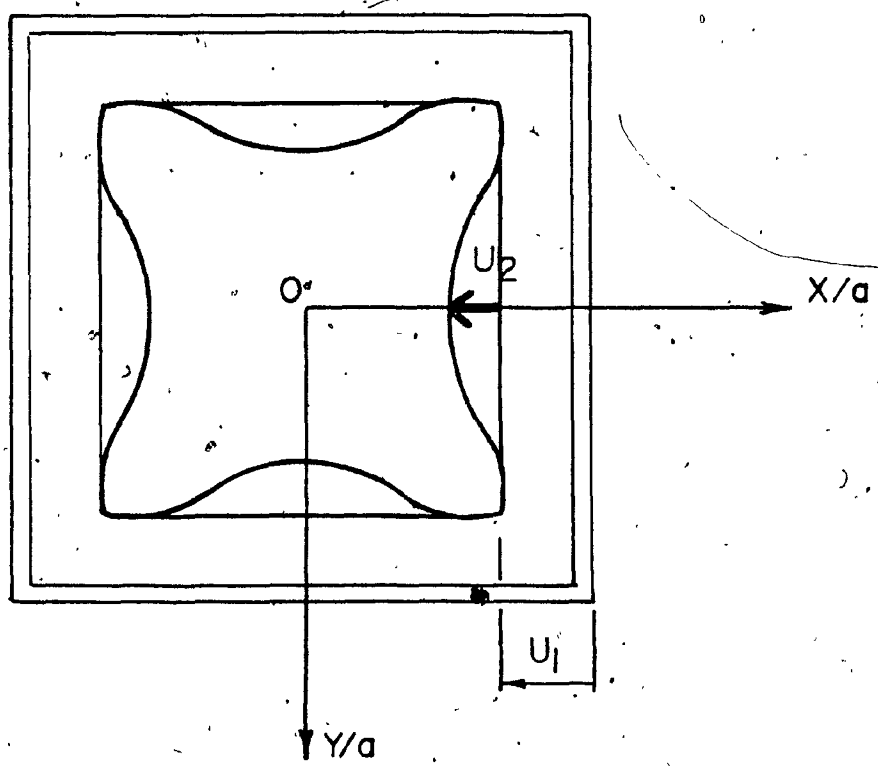


PLATE AFTER DEFORMATION

FIG (6.2)

6.2 VERTICAL DISPLACEMENT

The vertical displacement ω is given by equation (4.13)

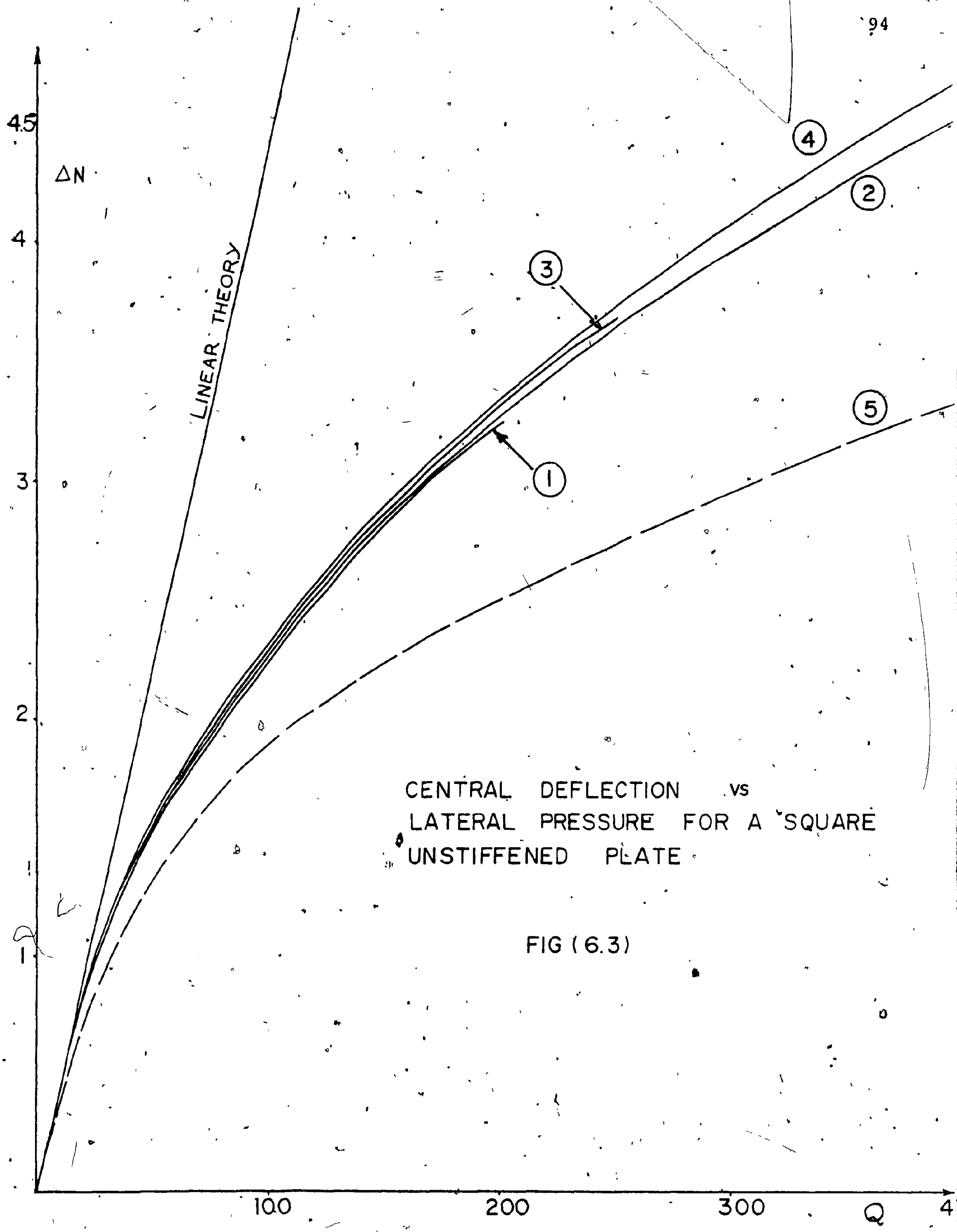
$$\omega = \delta_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} + \delta_2 \cos\frac{\pi x}{a} \cos\frac{3\pi y}{a} + \delta_2 \cos\frac{3\pi x}{a} \cos\frac{\pi y}{a} + \delta_3 \cos\frac{3\pi x}{a} \cos\frac{3\pi y}{a}$$

The values of δ_1, δ_2 and δ_3 were computed for ten intensities of the load, and for five values of the stiffener area, as before.

The central deflection is shown in Fig. (6.3), plotted against the intensity of loading for an unstiffened plate. Published theoretical solutions and experimental results for square isotropic plates are also shown in Fig. 6.3. The present solution was compared with Kaiser [5], Stipper [18], A. Schales and E.L. Bernstein [7], Brown and Harvey [2] and K.R. Rushton, [6].

The results obtained by Levy [10] for the case where the edges are kept straight and the shear stress and the resultant membrane tension are zero at the edges, are also plotted.

As shown, the present results obtained for the deflections in a square non-stiffened plate are in excellent agreement with the existing theoretical solutions and the experimental results of previous investigators. The maximum deviation (obtained at the maximum load considered) between the present



CENTRAL DEFLECTION vs LATERAL PRESSURE FOR A SQUARE UNSTIFFENED PLATE

FIG (6.3)

PUBLISHED & PRESENT RESULTS FOR SIMPLY SUPPORTED SQUARE PLATES

$\Delta_N = \Delta / t$ = CENTRAL DEFLECTION PARAMETER

$Q = q/E\,(a/t)^4$ = LOAD PARAMETER

① ——— KAISER — STIPPER — A. SCHOLES & E.L. BERNSTEN

② ——— PRESENT THEORY $\nu = 0.316$

③ ——— BROWN & HARVEY $\nu = 0.316$

④ ——— K. R. RUSHTON $\nu = 0.3$

⑤ ——— LEVY $\nu = 0.316$

solution and the dynamic relaxation method is about 5%. This difference may be due to different values of ν used. The departure from the linear theory is clearly shown in Fig. (6.3)

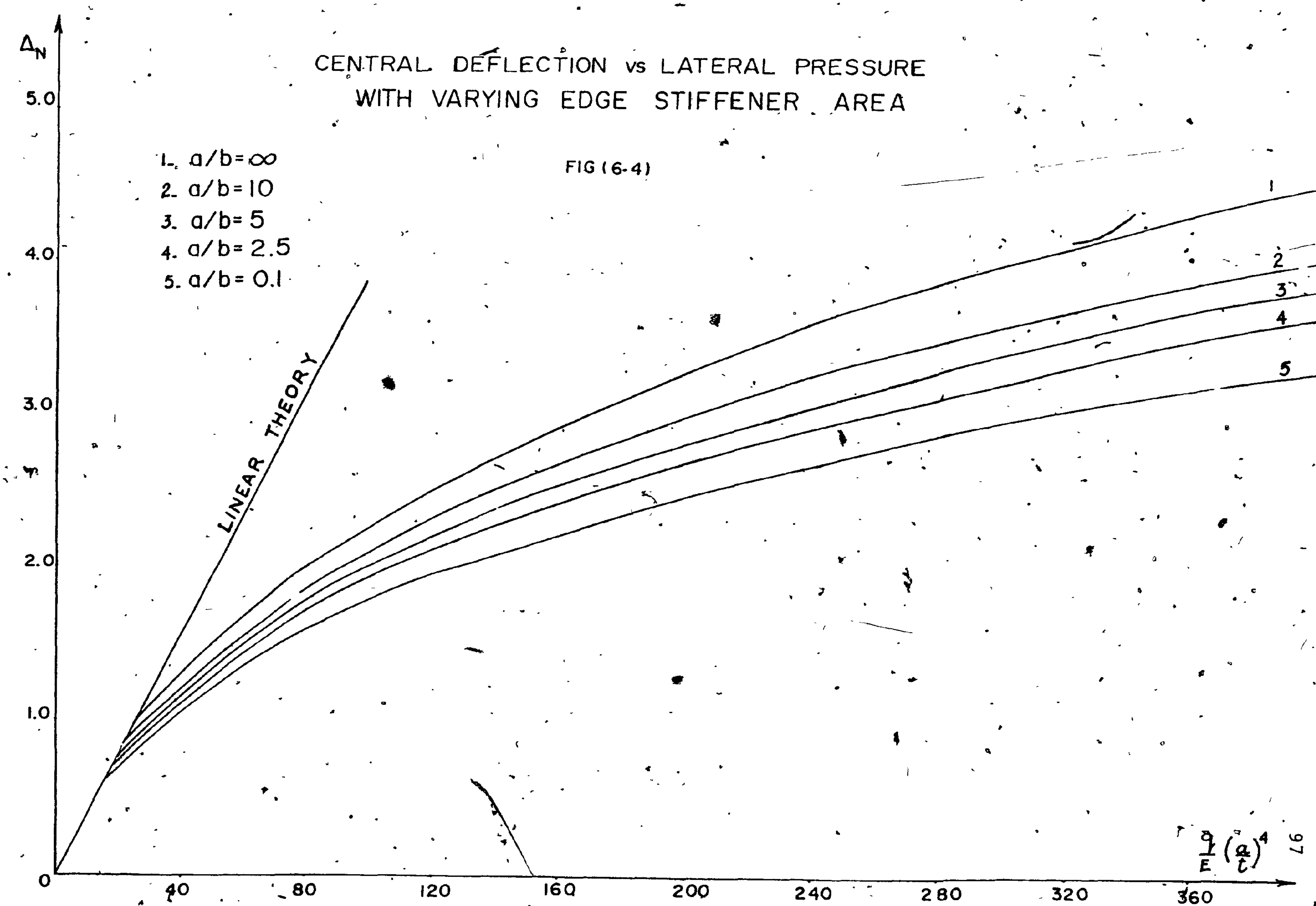
The central deflection, shown in Fig. (6.4), is plotted against the intensity of loading for five cases of stiffener area. As can be expected the effect of membrane stresses is least for case $a/b = \infty$, and increases as a/b decreases. There is no solution available for comparison for the values of $\infty > \frac{a}{b} > 0$.

The load-deflection graphs show how the central deflections predicted by the classical linear bending theory differ from those of the non-linear theory and how the difference increases rapidly with Q and slightly with b/a. It may be pointed out that the ratio of the deflection to plate thickness is an important index in the membrane action in a plate.

In Fig. (6.5), the case of $\frac{b}{a} \to \infty$ is compared with:

(a) The results obtained by Levy for the case where the edges are kept straight and the shear stress and the resultant membrane tension are zero at the edges.

(b) The results published by Bauer, Bauer, Becker and Reiss for the case of the B.C. of zero normal membrane stresses and zero tangential membrane displacement along the plate edges.



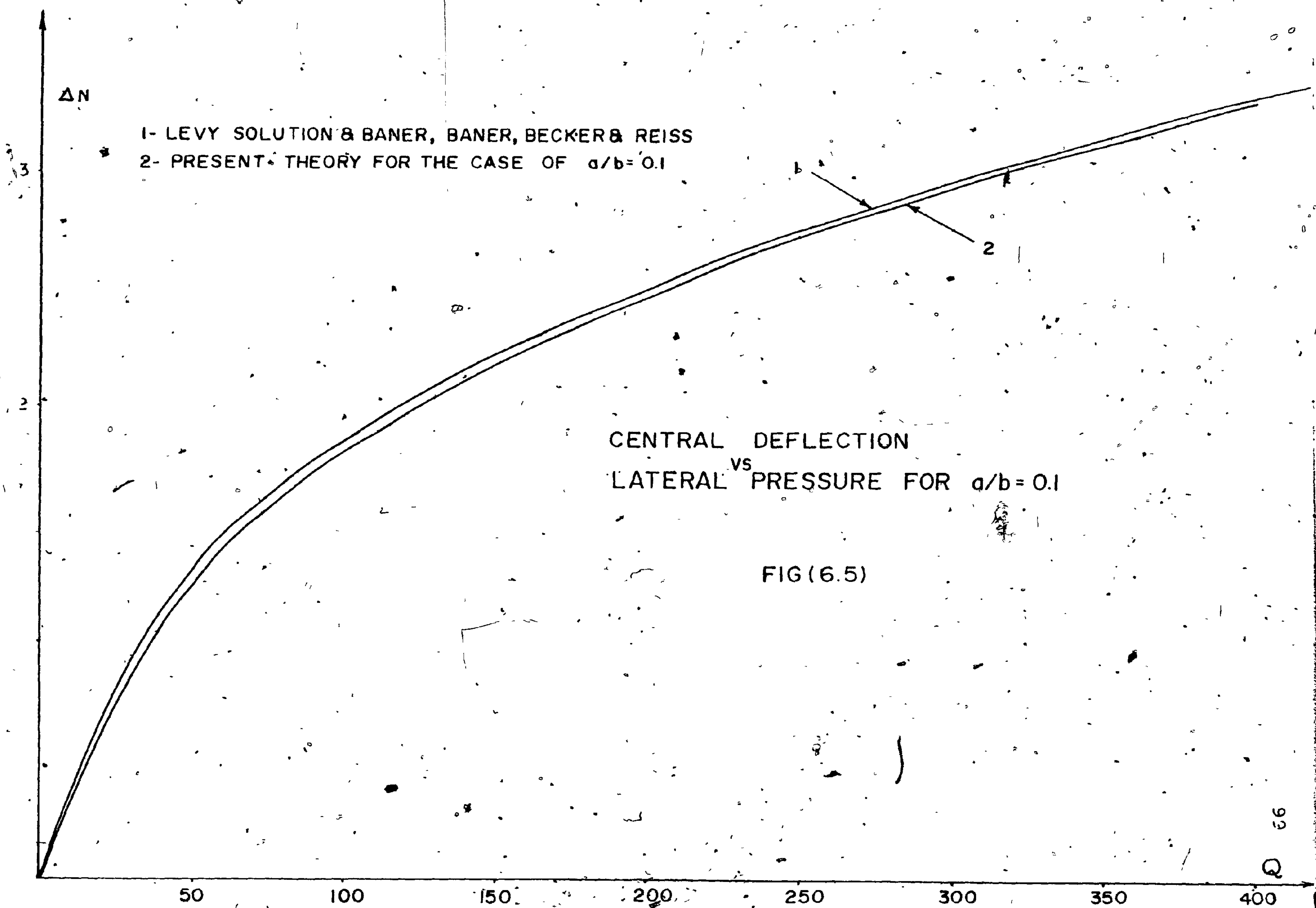
CENTRAL DEFLECTION vs LATERAL PRESSURE WITH VARYING EDGE STIFFENER AREA

FIG (6-4)

LEVY SOLUTION		PRESENT THEORY		
Q	LEVY Δ_N	Δ_N for $a/b=\infty$	Δ_N for $a/b=0.1$	Q
0	0.	1.306	1.110	40
12.1	0.486	2.016	1.614	80
29.4	0.962	2.522	1.962	120
56.9	1.424	2.928	2.237	160
99.4	1.87	3.272	2.469	200
161	2.307	3.573	2.671	240
247	2.742	3.843	2.852	280
358	3.174	4.088	3.016	320
497	3.600	4.313	3.166	360
		4.523	3.306	400

TABLE (6.2.1)

TABLE (6.2.2)



CENTRAL DEFLECTION vs LATERAL PRESSURE FOR $a/b = 0.1$

FIG (6.5)

As seen from Fig. (6.5) the agreements are excellent, even at the maximum load considered.

The effect of varying the stiffener area on the central deflection is shown in Fig. (6.6). This figure shows the ratio of the central deflection to that corresponding to stiffeners with infinite area. The ratio to some extent depends upon the intensity of the loading, and when the stiffener area is zero, all the curves shown in the figure have a maximum gradient. When b/a = .4, the plate is virtually fully restraint, the deflection at the plate center for this case being within 10% of the fully restraint value.

Presentative graphs of the variation with x at y = 0 and along the diagonal of the vertical displacements at five load levels, are presented in Figures (6.7) and (6.8). The vertical displacement attains its maximum at the center of the plate for all loads considered. The plate flattens slightly at the center for larger loads, and the flattening effect increases as q increases.

6.3 MEMBRANE STRESSES

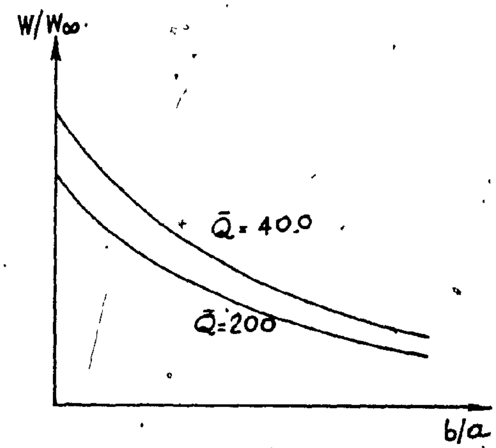
The membrane stresses in the x and y directions are given by:

$$\sigma_{xm} = \frac{E\pi}{(1-\nu^2)} [(1-\nu^2)C_1 + (1+\nu)C_3 \quad \cos\frac{\pi y}{a}] \cos\frac{\pi x}{a} \tag{6.1}$$

EFFECT OF VARYING THE STIFFENER AREA ON THE CENTRAL DEFLECTION

Q = 200

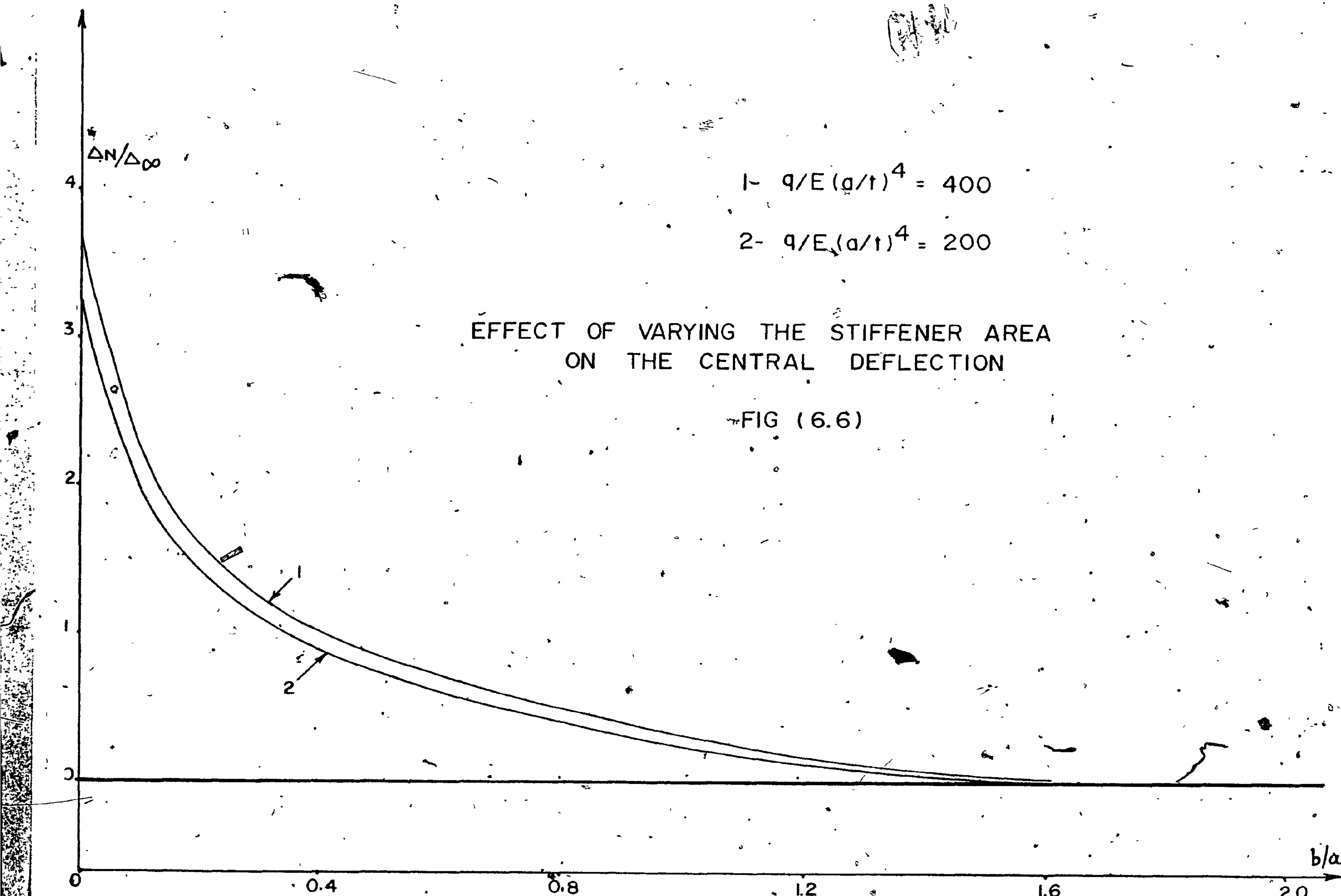
b/a	W	W/W∞
0	3.2722	1.3250
0.1	2.9603	1.1987
0.2	2.8233	1.1432
0.4	2.6951	1.0914
10.56	2.4695	1



Q = 400

b/a	W	W/W∞
0	4.5231	1.368
0.1	4.0385	1.2213
0.2	3.8306	1.1585
0.4	3.6389	1.1005
10.56	3.3066	1

TABLE (6.3)

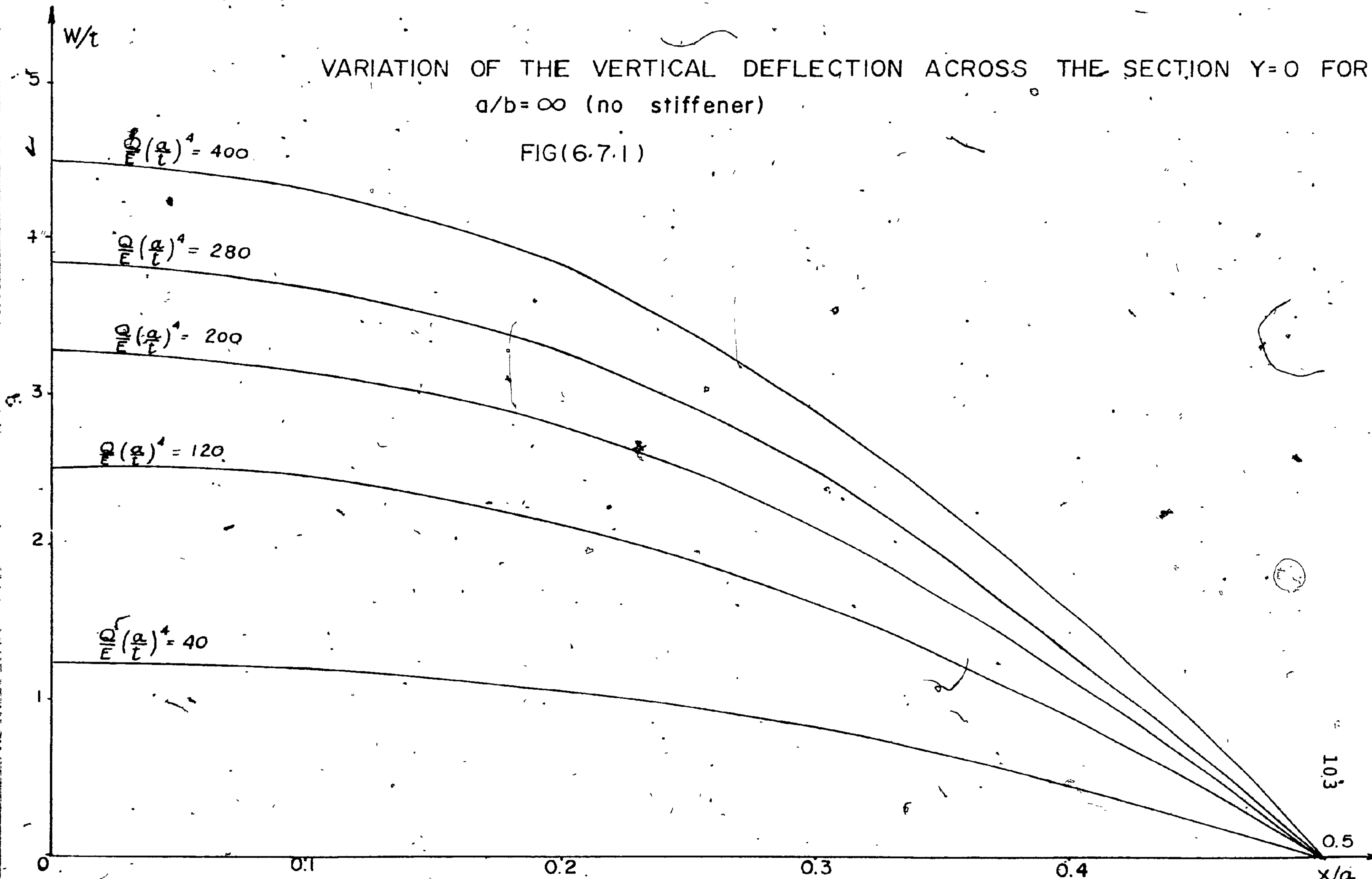


EFFECT OF VARYING THE STIFFENER AREA ON THE CENTRAL DEFLECTION

FIG (6.6)

VARIATION OF THE VERTICAL DEFLECTION ACROSS THE SECTION Y=0 FOR $a/b=\infty$ (no stiffener)

FIG(6.7.1)



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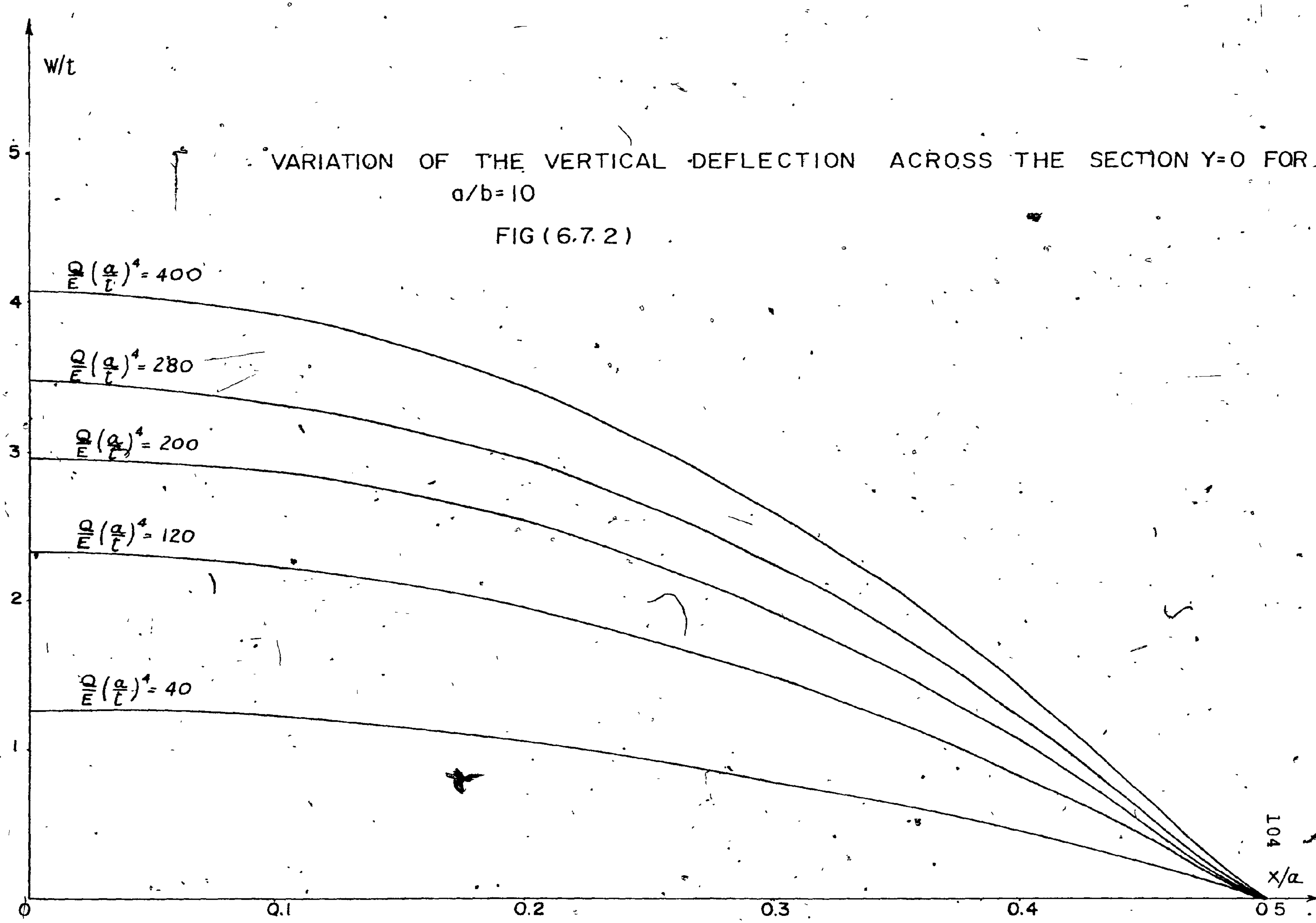
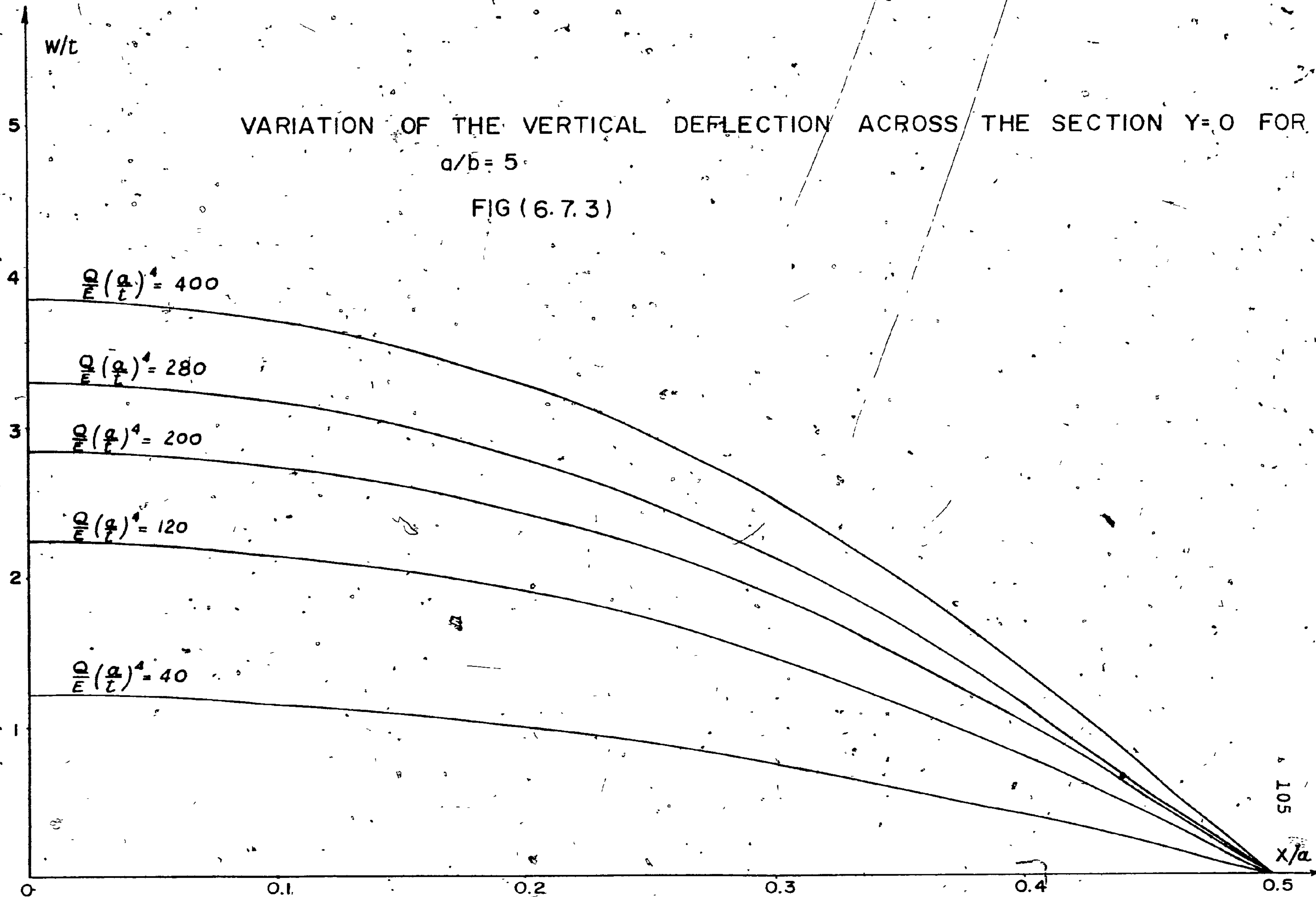


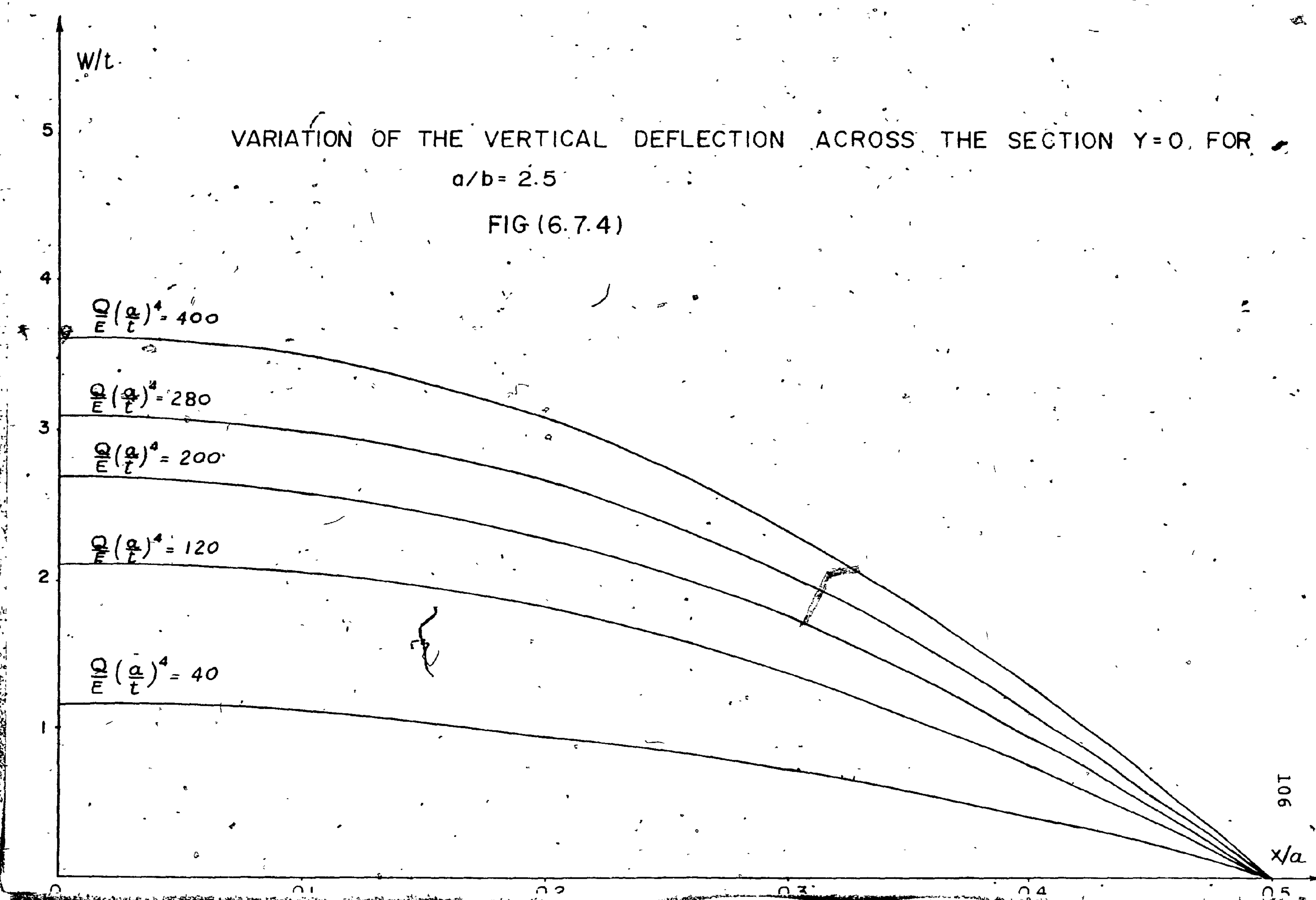
VARIATION OF THE VERTICAL DEFLECTION ACROSS THE SECTION Y=0 FOR a/b=10

FIG (6.7.2)



VARIATION OF THE VERTICAL DEFLECTION ACROSS THE SECTION Y=0 FOR a/b = 5

FIG (6.7.3)



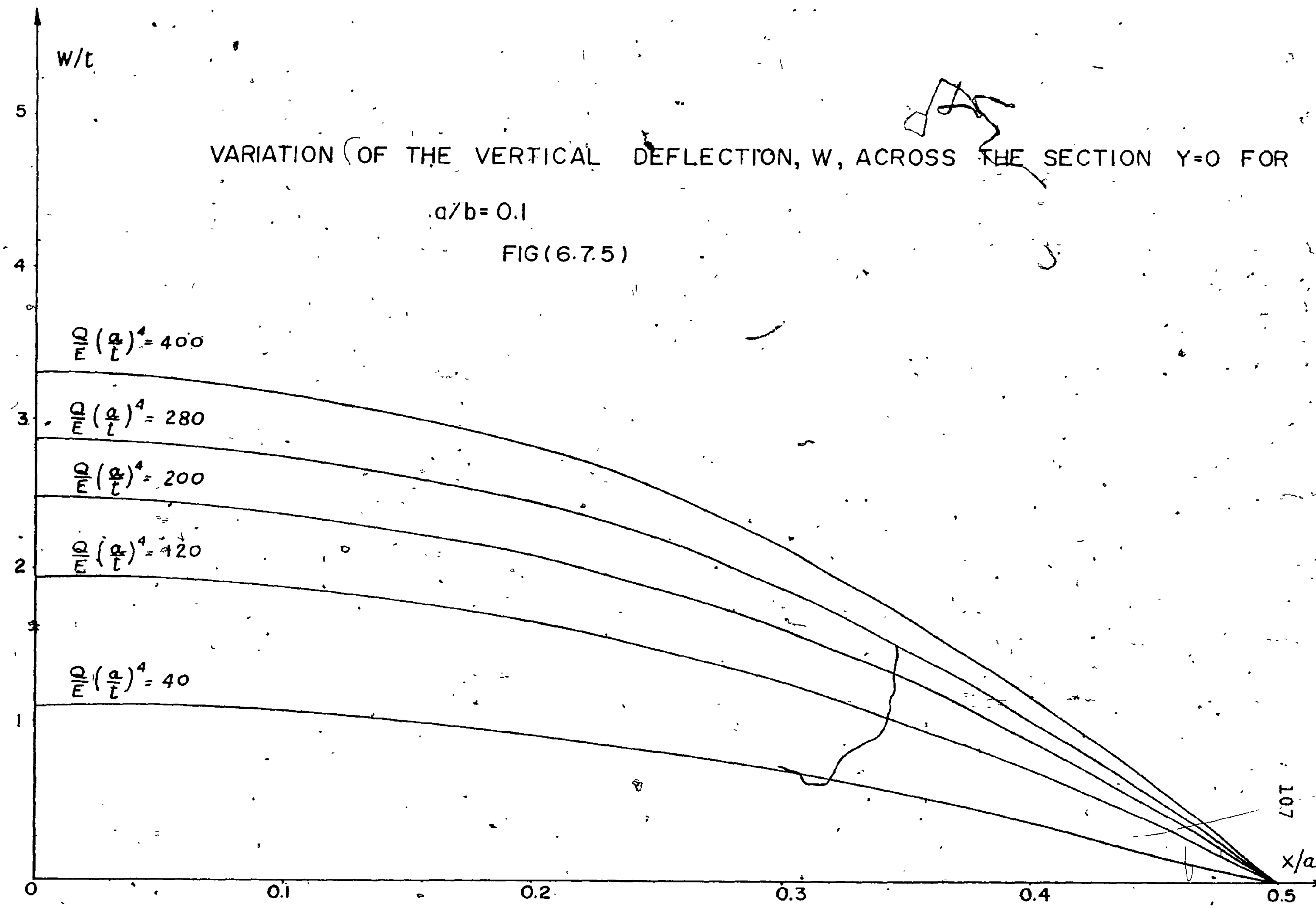
VARIATION OF THE VERTICAL DEFLECTION ACROSS THE SECTION Y=0 FOR $a/b = 2.5$

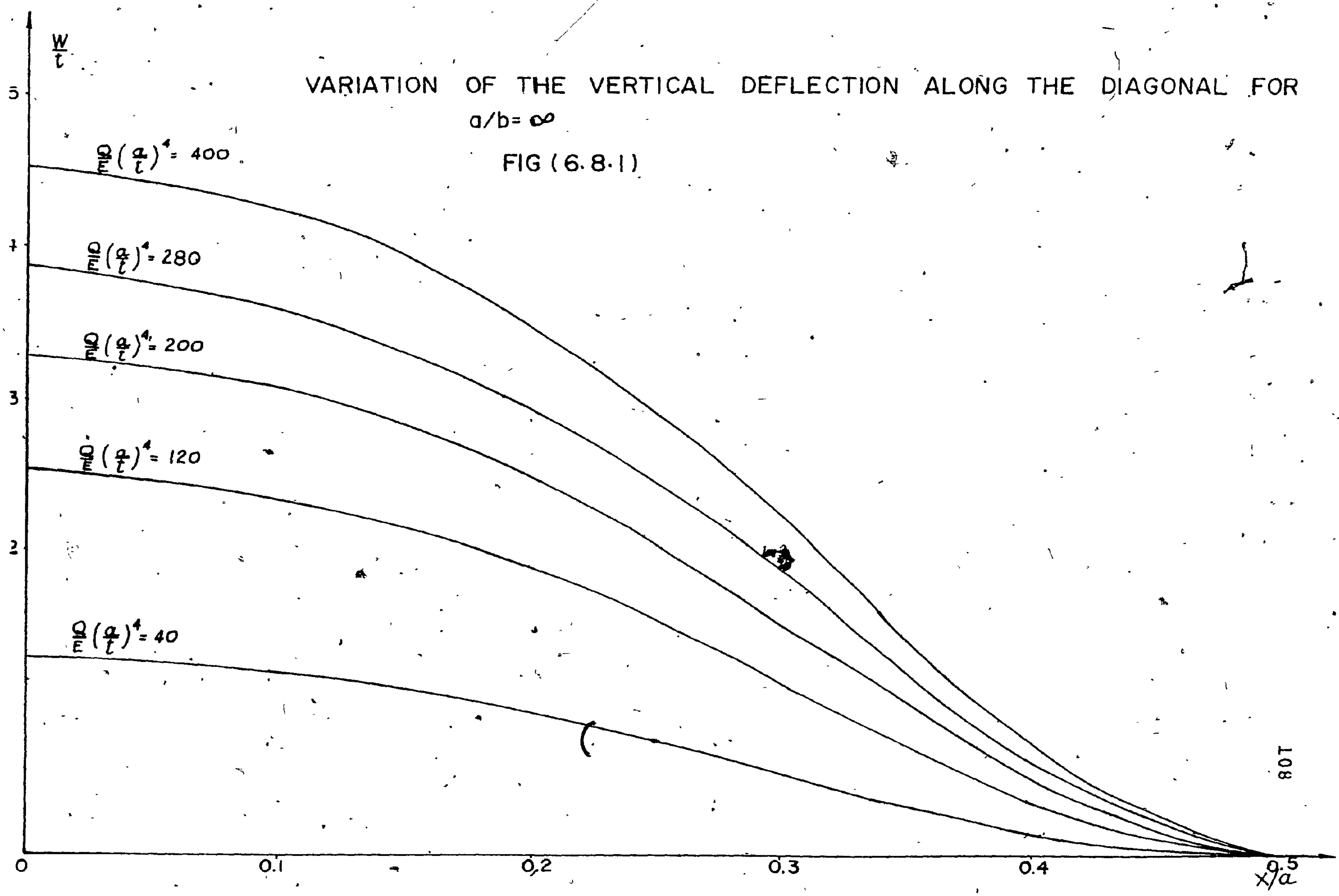
FIG (6.7.4)

VARIATION OF THE VERTICAL DEFLECTION, W, ACROSS THE SECTION Y=O FOR

a/b= 0.1

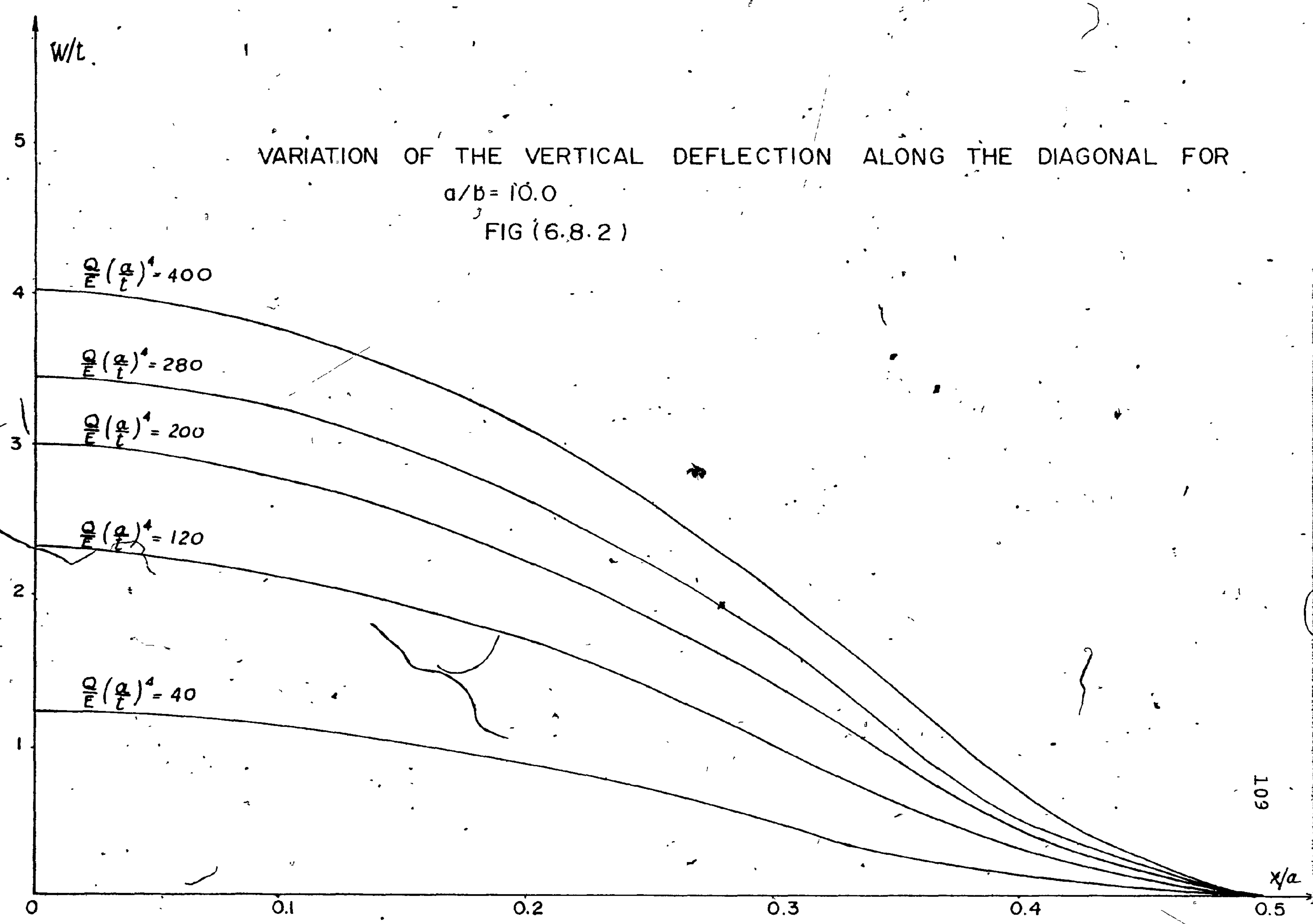
FIG (6.7.5)





VARIATION OF THE VERTICAL DEFLECTION ALONG THE DIAGONAL FOR $a/b = \infty$

FIG (6.8.1)



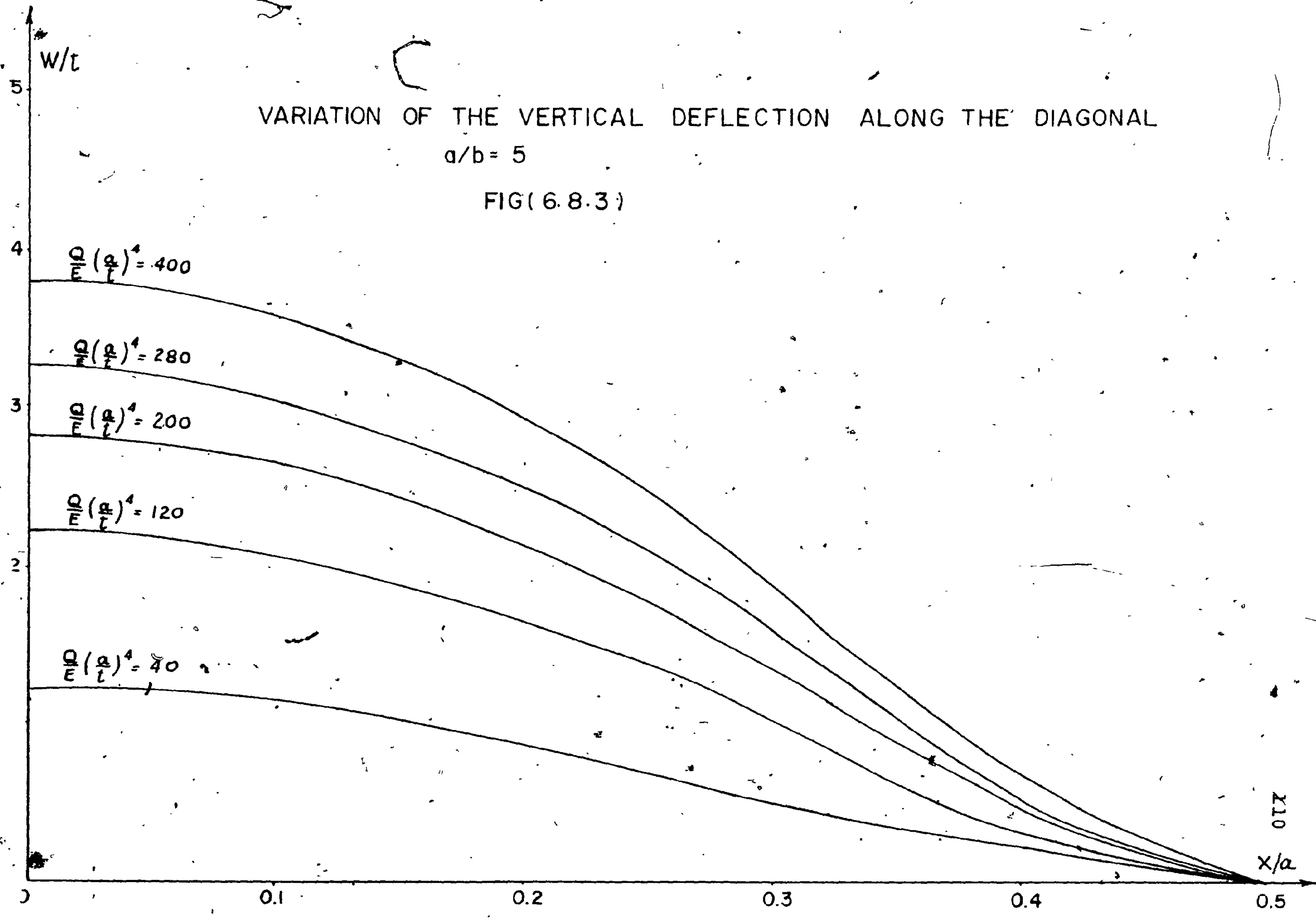
VARIATION OF THE VERTICAL DEFLECTION ALONG THE DIAGONAL FOR

$a/b = 10.0$

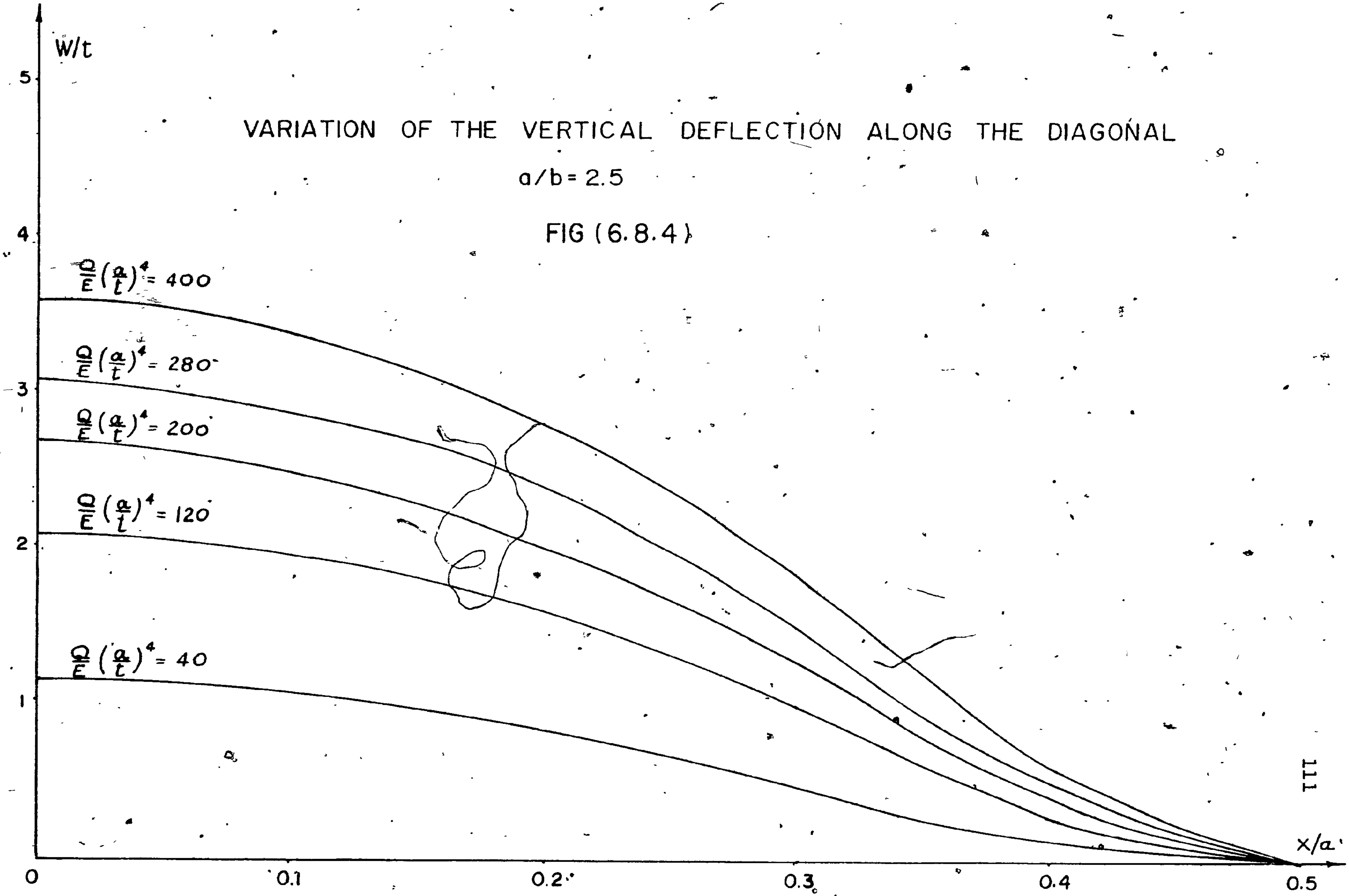
FIG (6.8.2)

VARIATION OF THE VERTICAL DEFLECTION ALONG THE DIAGONAL

a/b = 5



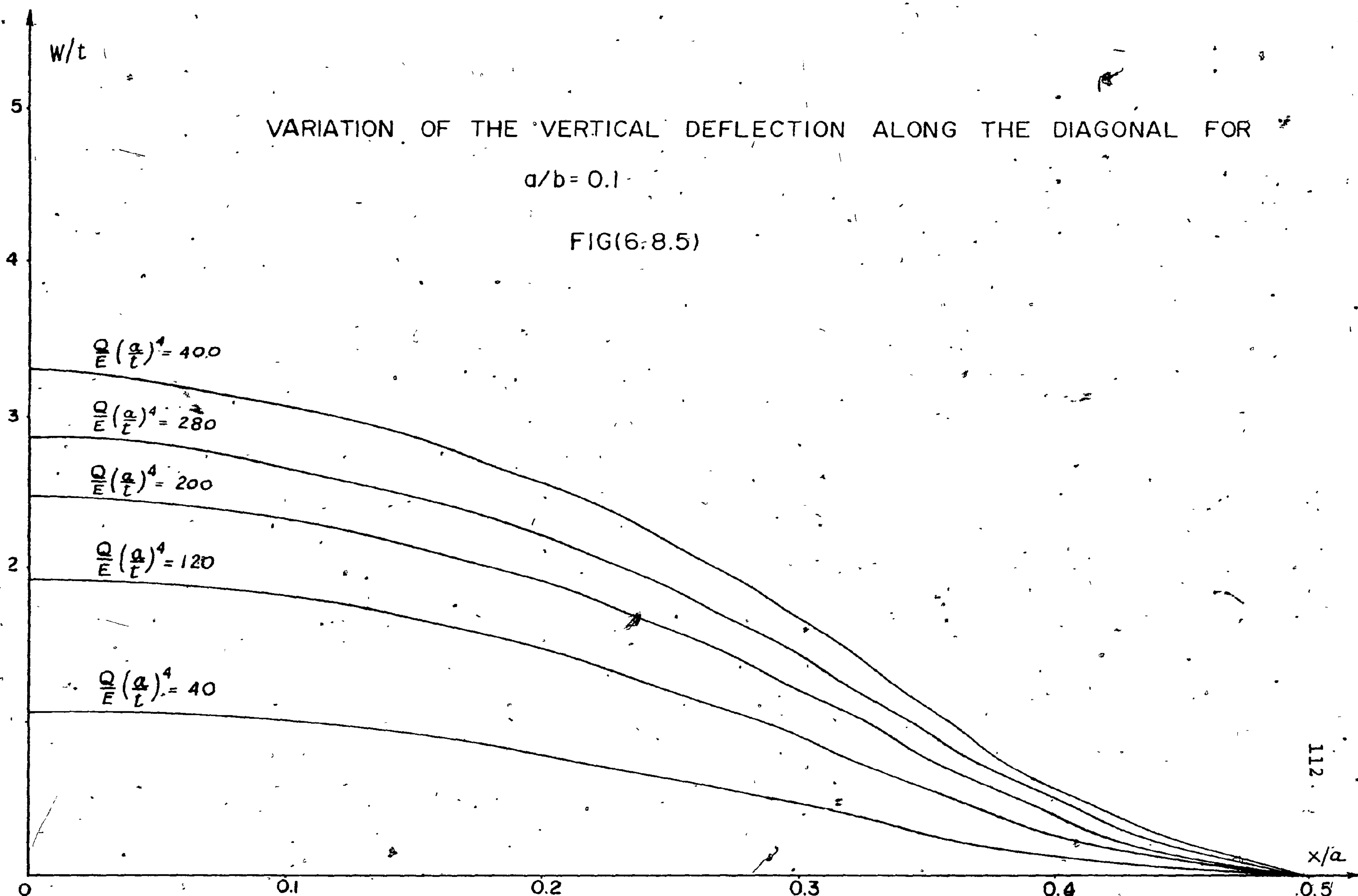
FIG(6.8.3)



VARIATION OF THE VERTICAL DEFLECTION ALONG THE DIAGONAL

a/b = 2.5

FIG (6.8.4)



VARIATION OF THE VERTICAL DEFLECTION ALONG THE DIAGONAL FOR

$a/b = 0.1$

FIG(6.8.5)

$$\sigma_{ym} = \frac{E}{(1-\nu^2)} [(1-\nu^2)C_1 + (1+\nu)C_3 \quad \cos\frac{\pi x}{a}] \cos \frac{\pi y}{a} \qquad (6.2)$$

which indicates that:

(a) $\sigma_{xm} = 0$ at $x = \pm \frac{a}{2}$

(b) $\sigma_{ym} = 0$ at $y = \pm \frac{a}{2}$

Across the section at $y = 0$

$$\sigma_{xm} = \frac{E\pi}{(1-\nu^2)} [(1-\nu^2)C_1 + (1+\nu)C_3] \cos\frac{\pi x}{a}$$

$$\frac{\sigma_{xm}}{E} = \frac{\pi}{1-\nu^2} [(1-\nu^2)C_1 + (1+\nu)C_3] \cos\frac{\pi x}{a} \qquad (6.3)$$

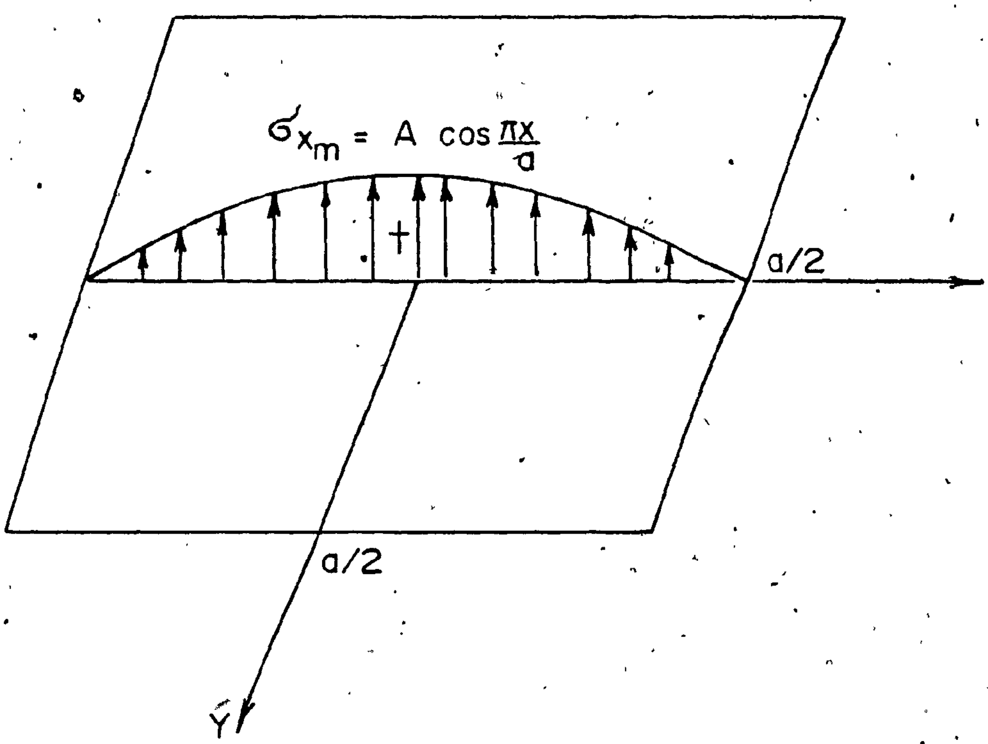
C_1 always has a negative value, and

C_3 always has a positive value, but

C_3 is always greater than C_1 in absolute value.

This indicates that the distribution of the membrane stresses across the section $y = 0$ is always in tension, having the cosine shape with zero value at the edges and a maximum value at the plate center. (see Fig. 6.9).

The membrane tension at the center of the plate is plotted against the load Q in Fig.(6.10)for five different



VARIATION OF σ_{x_m} ACROSS THE SECTION Y=0

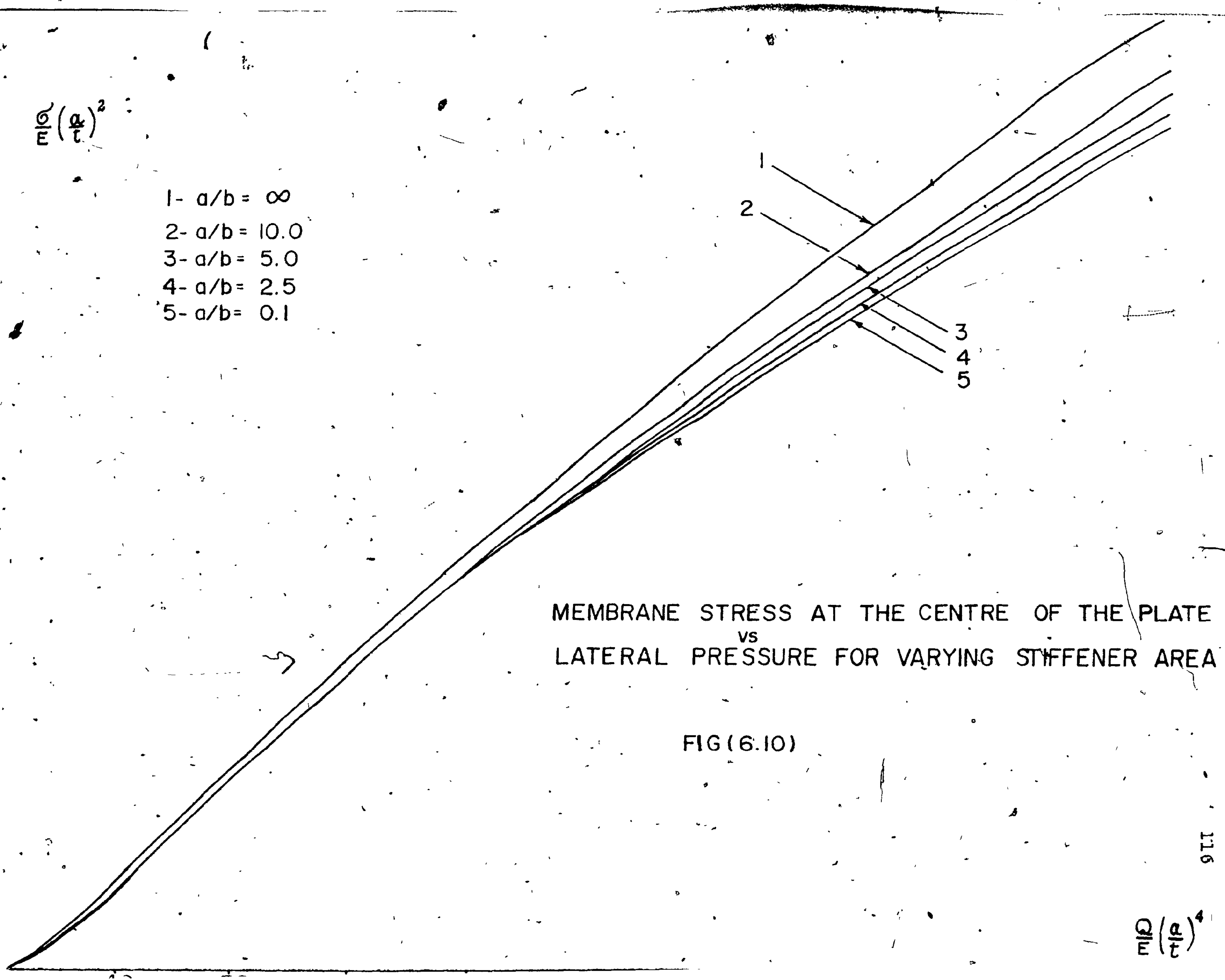
FIG (6.9)

values of stiffener area.

For unstiffened plates, a comparison has been made between the present solution and with Brown and Harvey, and K.R. Rushton solutions. At Kaiser load ($Q \approx 120$), the present theory shows a value of 13% higher than those given by the earlier investigations.

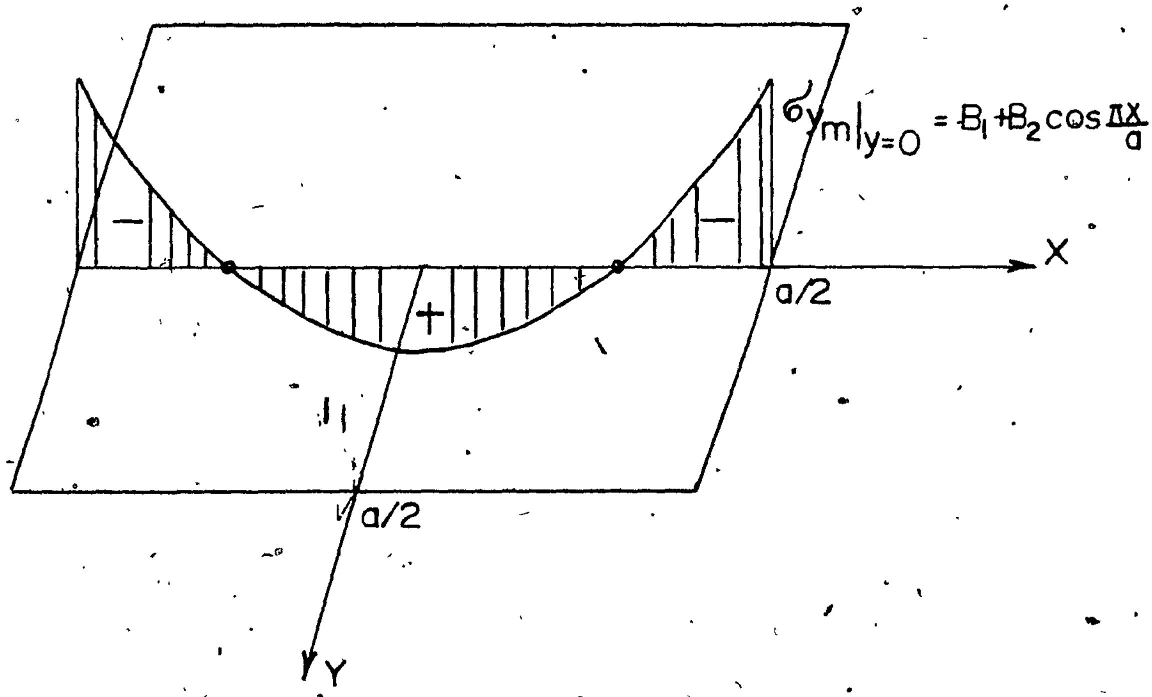
The effect of varying the stiffener area on the membrane tension at the plate center, is also shown in Fig. (6.10). Up to load level ($Q = 160$), the variations in the stiffener area have practically no effect on the membrane tension stresses. As the load increases above $Q = 160$, the membrane stresses decrease slightly, as the stiffener area increases. This behaviour can be explained as follows. The resultant total membrane forces normal to any longitudinal or transverse section of the plate must be zero, the tensile stresses in the centre of the section being counter-balanced by the compressive stresses near the edges. As indicated in Section (6.4), the neutral line moves towards the plate edges with an increase of the stiffener area, thus increasing the central area carrying tension. To maintain equilibrium, the maximum value of the membrane tension stress at the plate center decreases (see Fig. 6.11.1.)

The membrane stresses in the x-direction across the section $x = 0$ is given by:



MEMBRANE STRESS AT THE CENTRE OF THE PLATE
vs
LATERAL PRESSURE FOR VARYING STIFFENER AREA

FIG (6.10)



VARIATION OF σ_{y_m} ACROSS THE SECTION Y=0

FIG (6.11.1)

VARIATION OF σ_{y_m} ACROSS THE CENTRE LINES

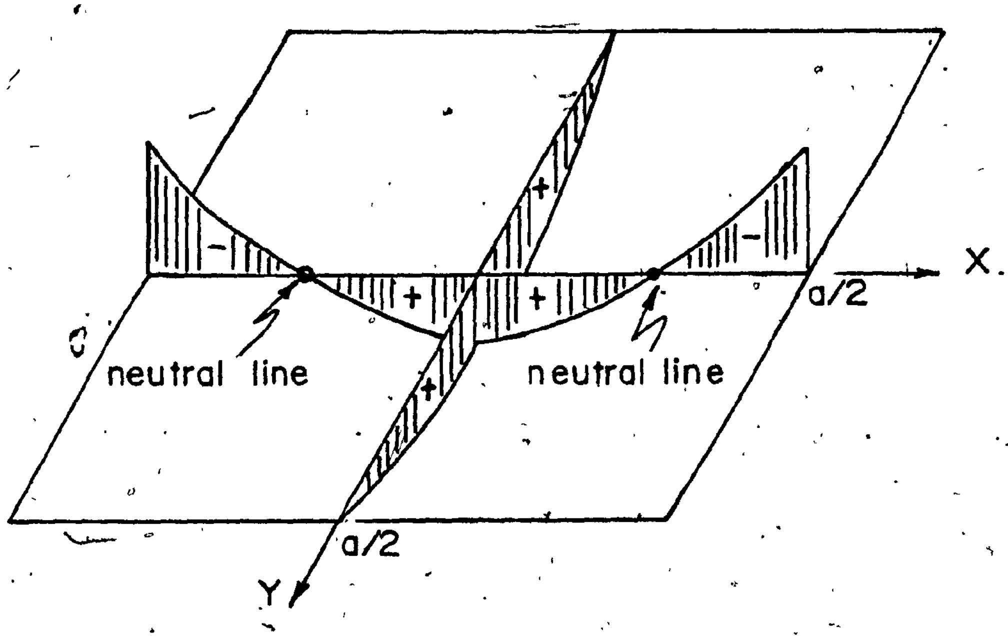


FIG (6.11.2)

$$\sigma_{xm}\Big|_{x=o} = \frac{E\pi}{1-\nu^2}\left[(1-\nu^2)C_1 + (1+\nu)C_3 \cos\frac{\pi y}{a}\right] \tag{6.4}$$

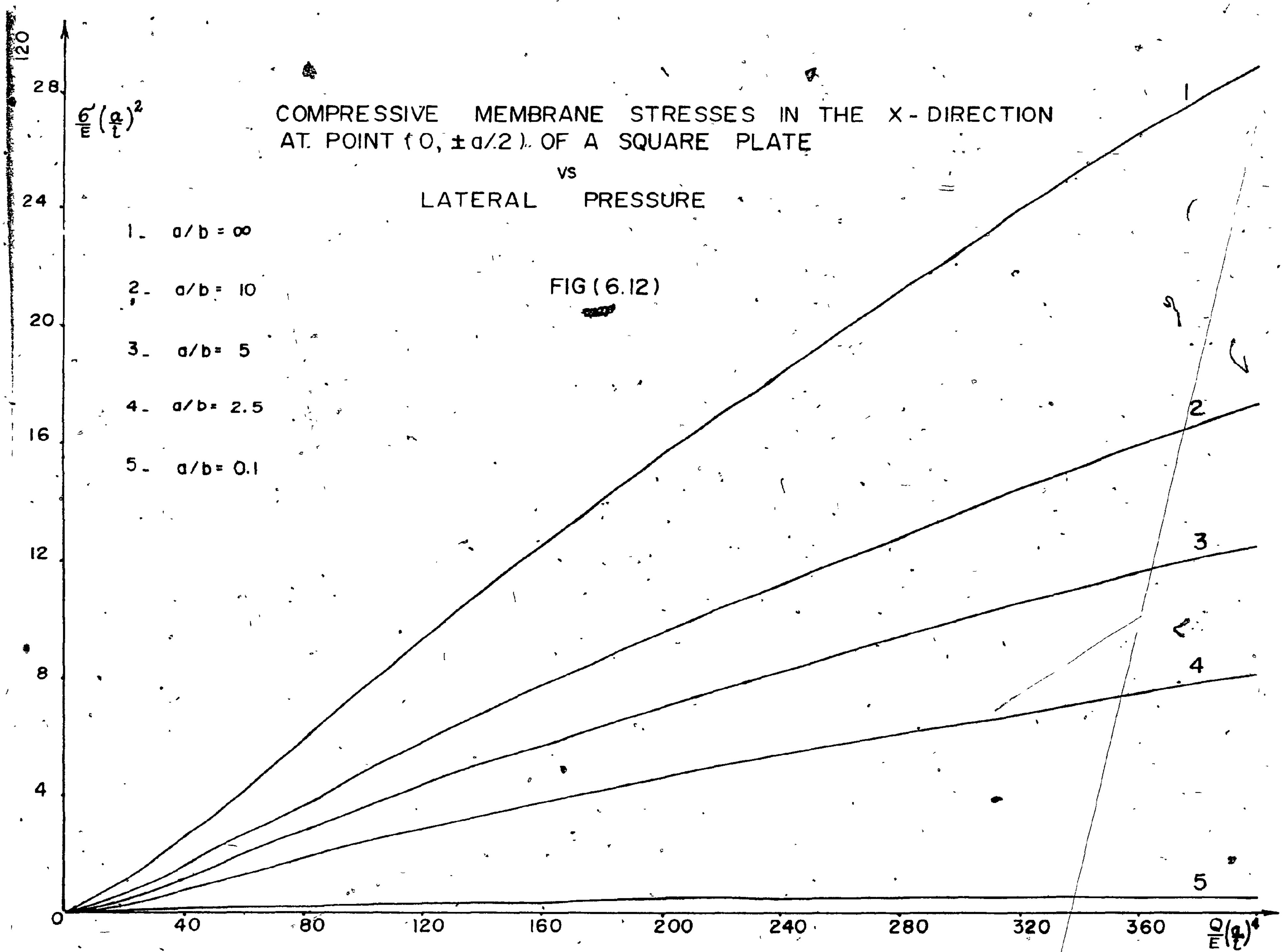
This relation is plotted as shown in Fig. 6.11.

The membrane compressive stress at the mid-points of the edges of the plate, in a direction parallel to the edges, is plotted against the pressure Q, in Fig. (6.12) for five different values of stiffener areas.

It may be noted that the in-plane compressive stress caused by membrane action at the mid-points of the edges, in a direction parallel to the edges for a plate with no stiffeners, is greater than the tensile membrane stress at the center of the plate.

The effect of varying the stiffener area on the value of the compressive stress at the mid-points of the plate edges is also shown in Fig. (6.12). As seen from this figure, the compressive stress decreases rapidly with the increase in the stiffener area.

As the plate is free from any in-plane external forces, the resultant membrane forces normal to any longitudinal or transverse section of the plate, must be zero, the tensile stress in the central area of the section being counter-balanced by the compressive stress near the edges. At the



COMPRESSIVE MEMBRANE STRESSES IN THE X-DIRECTION
AT POINT (0, ±a/2) OF A SQUARE PLATE
VS
LATERAL PRESSURE
1_ a/b = ∞
2_ a/b = 10
3_ a/b = 5
4_ a/b = 2.5
5_ a/b = 0.1
$\frac{\sigma}{E}\left(\frac{a}{t}\right)^2$
$\frac{Q}{E}\left(\frac{a}{t}\right)^4$
0
4
8
12
16
20
24
28
40
80
120
160
200
240
280
320
360
1
2
3
4
5

FIG (6.12)

section $x = 0$ the total force is given by:

$$\int_0^{a/2} t\,\sigma_x\Big|_{x=0} dy = \frac{E\pi t}{1-\nu^2}\Big[(1-\nu^2)C_1.b + \int_0^{a/2}(1-\nu^2)C_1 dy +$$

$$+ \int_0^{a/2}(1+\nu)C_3 \cos\frac{\pi y}{a}\,dy\Big] =$$

$$= \frac{E\pi ta}{1-\nu^2}\Big[\frac{(1-\nu^2)}{a/b}C_1 + \frac{(1-\nu^2)}{2}C_1 + \Big(\frac{1+\nu}{\pi}\Big)C_3\Big] \qquad (6.5)$$

This resultant total membrane force across the centre cross-section is given with the computer output. It never exceeded 0.6%, and was in most cases around 0.2%, of the tensile membrane force across the central zone of the plate, which provides a check on the numerical accuracy of the solution.

6.4 NEUTRAL LINE

The distribution of the membrane stresses normal to any longitudinal or transverse section of the plate has the shape given in Fig.(6.11). To find the locus of all the points of the plate having zero membrane stresses (neutral curve) we equate Equations(6.1) and(6.2) to zero.

$$\sigma_{xm} = 0 = \frac{E\pi}{1-\nu^2}\cos\frac{\pi x}{a}\Big[(1-\nu^2)C_1 + (1+\nu)C_3 \cos\frac{\pi y}{a}\Big]$$

$$\cos\frac{\pi x}{a} = 0 \qquad\qquad x/a = \pm .5$$

$$(1-\nu^2)C_1 + (1+\nu)C_3\cos\frac{\pi y}{a} = 0 \qquad \pm\frac{y}{a} = \cos^{-1} - \frac{(1-\nu^2)}{(1+\nu)}\frac{C_1}{C_3} \qquad (6.6)$$

Similarly from $\sigma_{ym} = 0$

$$\frac{y}{a} = \pm .5 \quad \text{and} \quad \pm \frac{x}{a} = \cos^{-1} - \frac{(1-\nu^2)}{(1+\nu)} \frac{C_1}{C_3} \tag{6.7}$$

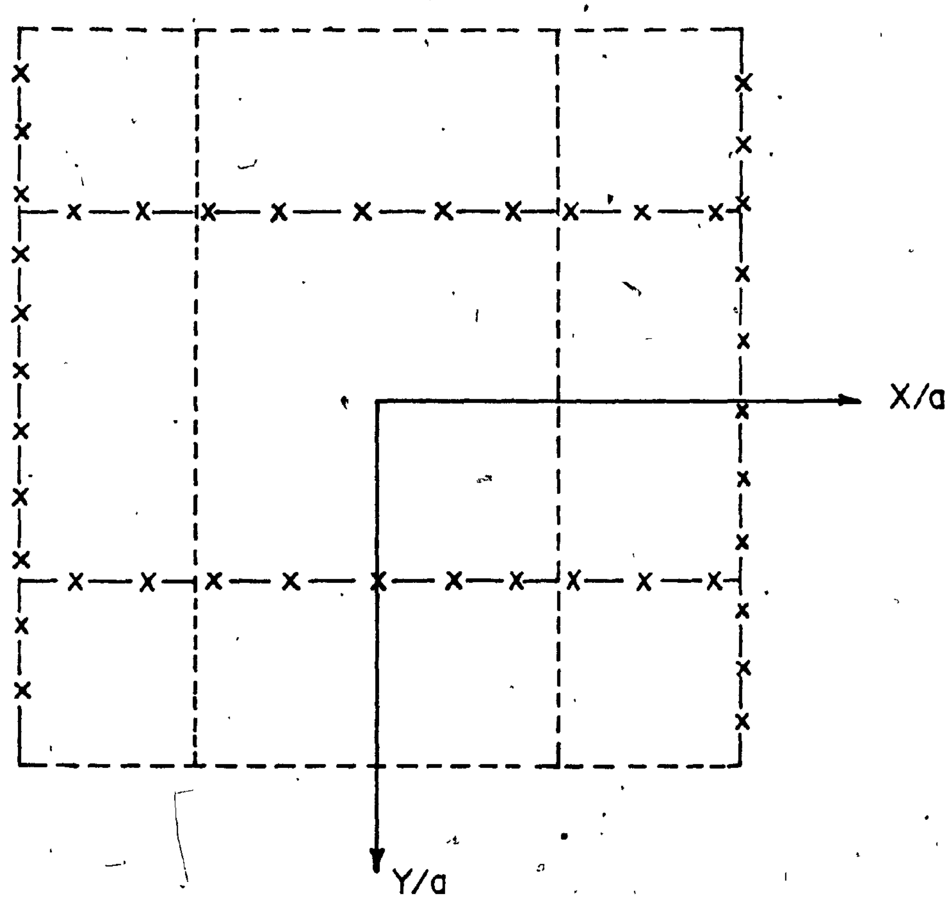
Equations (6.6) and (6.7) indicate that the neutral curve is a straight line, see Fig. (6.13).

The location of the neutral line is computed for each step of loading and each case of stiffener areas. The computer output indicates:

1) The location of the neutral line is almost the same for all steps of loading, i.e., the location of the neutral line does not change with the change of the load level.

2) The neutral lines move towards the plate edges, as the stiffener area increases. This is clearly shown in Fig. (6.14). In fact, as the stiffeners area approaches infinity, the neutral line coincides with the plate edges, thus dividing the system into a whole plate loaded with membrane tension and a stiffener loaded with axial compression. See Figure (6.15).

LOCUS OF THE NEUTRAL LINE



—x—x— lines of zero σ_{x_m}

- - - - - - - lines of zero σ_{y_m}

FIG (6.13)

a/b	x/a
∞	$\pm$ 0.2809
10	$\pm$ 0.3223
5	$\pm$ 0.3499
2.5	$\pm$ 0.3851
0.1	$\pm$ 0.4908

VARIATION OF THE NEUTRAL LINE WITH a/b, AT $Q/E \cdot (a/t)^4 = 400$

TABLE (6.4)

VARIATION OF THE NEUTRAL LINE WITH a/b , AT Q = 400

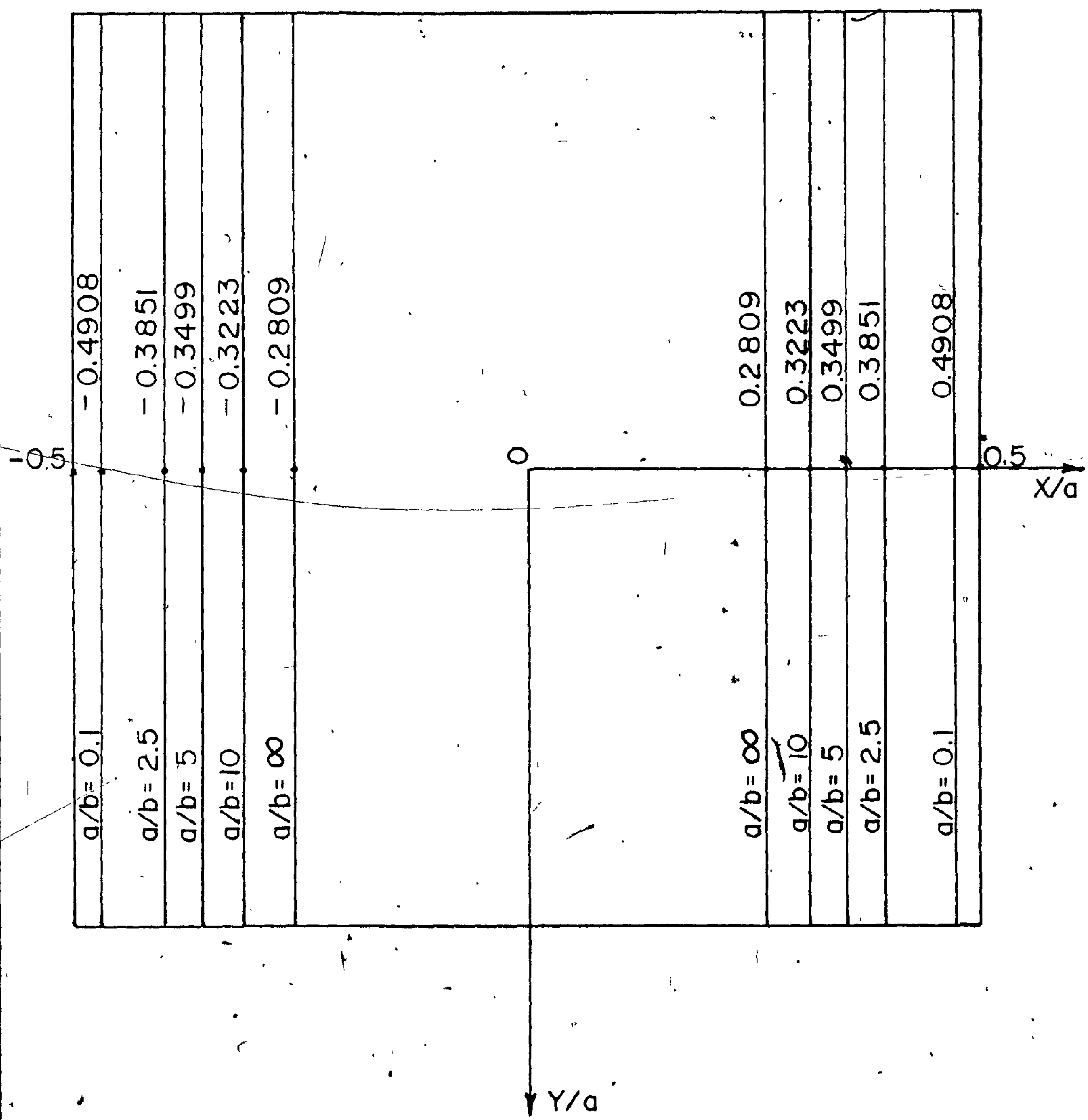
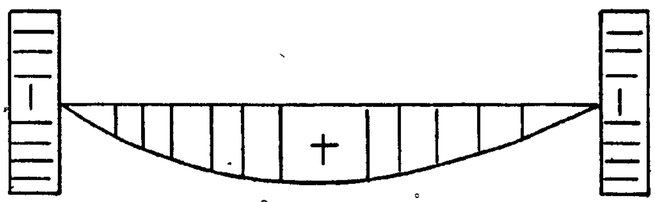


FIG (6.14)



DISTRIBUTION OF MEMBRANE STRESSES
NORMAL TO SECTION Y = 0, FOR THE ULTIMATE STAGE
AS $A_s \to \infty$

FIG (6.15)

6.5 MEMBRANE SHEAR STRESSES

The distribution of the membrane shear stresses in the plate is given by:

$$\tau_{xym} = \frac{E\pi^2}{1-\nu^2} (\sin\frac{\pi x}{a}) [(1-\nu^2) C_1 (\frac{y}{a}) + (\frac{1+\nu}{\pi}) C_3 \sin\frac{\pi y}{a}] \tag{6.8}$$

at $x = \pm \frac{a}{2}$

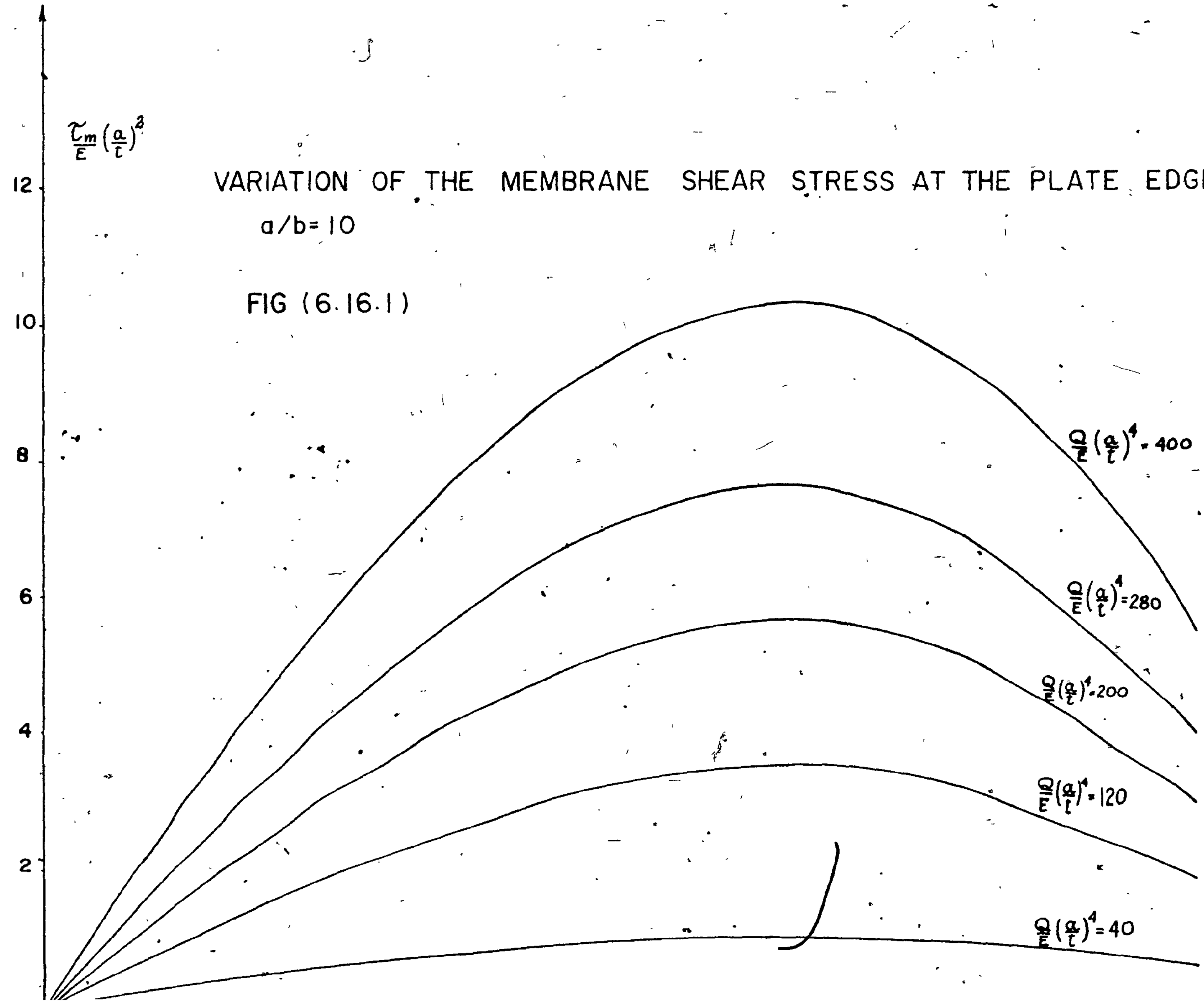
$$\tau_{xym}\Big|_{x=\pm a/2} = \frac{E\pi^2}{(1-\nu^2)} \{(1-\nu^2) C_1 (\frac{y}{a}) + (\frac{1+\nu}{\pi}) C_3 \sin\frac{\pi y}{a}\} \tag{6.9}$$

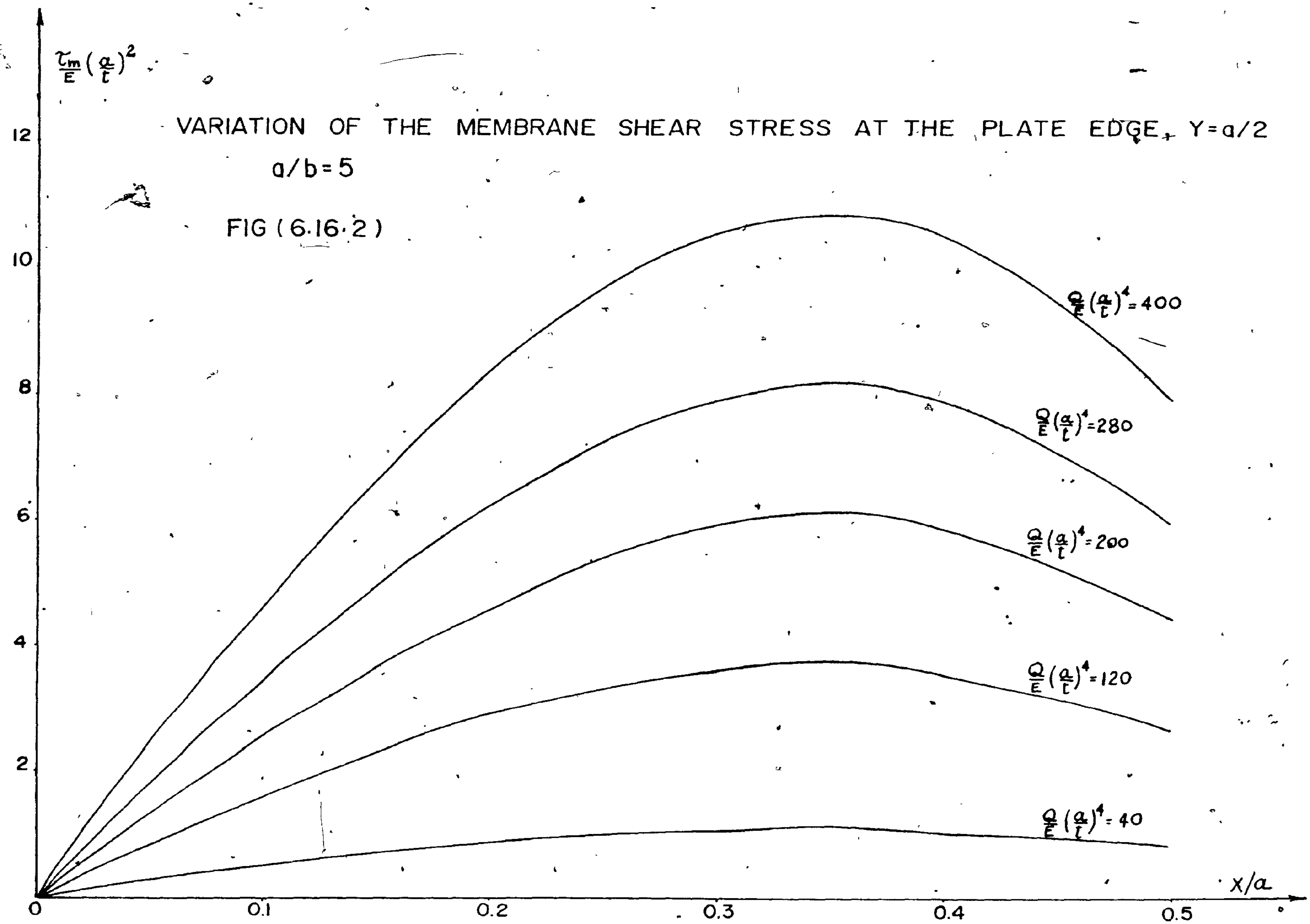
The variation of the membrane shear stress at the plate edge for five steps of loading and for four cases of different stiffener's area, is plotted in Fig.(6.16). As shown in Fig.(6.16) ,the shearing stress pattern starts from zero at the mid-edges, increasing rapidly until it reaches a maximum at the neutral line, then decreasing slowly. As the stiffener area increases, and for the same load, the shearing stress at the edges becomes higher. As the stiffener area increases, the location of the maximum shear stress moves towards the plate corner, until it coincides with the corner as the stiffener area approaches infinity.

VARIATION OF THE MEMBRANE SHEAR STRESS AT THE PLATE EDGE, Y= a/2

a/b= 10

FIG (6.16.1)

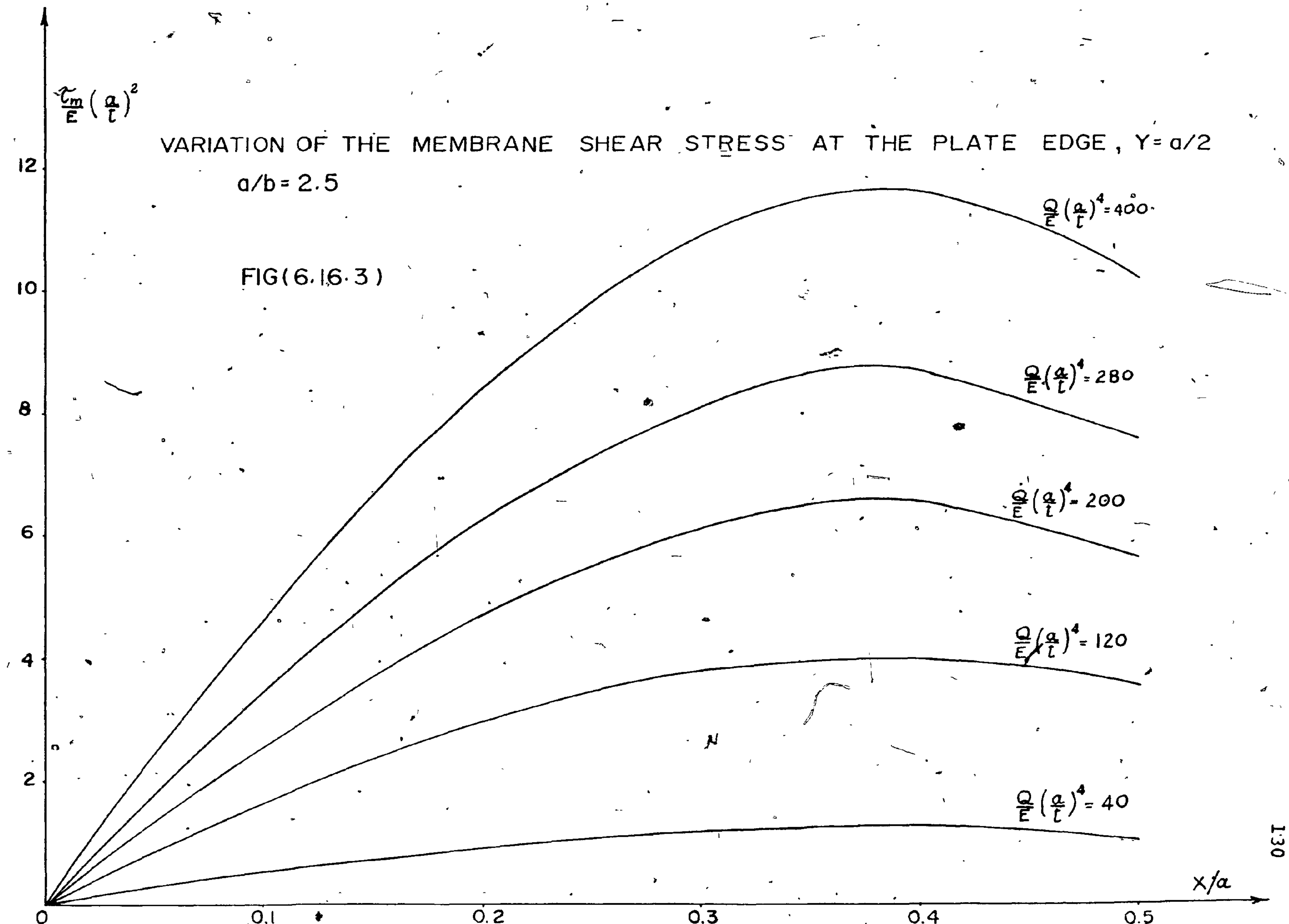




VARIATION OF THE MEMBRANE SHEAR STRESS AT THE PLATE EDGE, Y = a/2

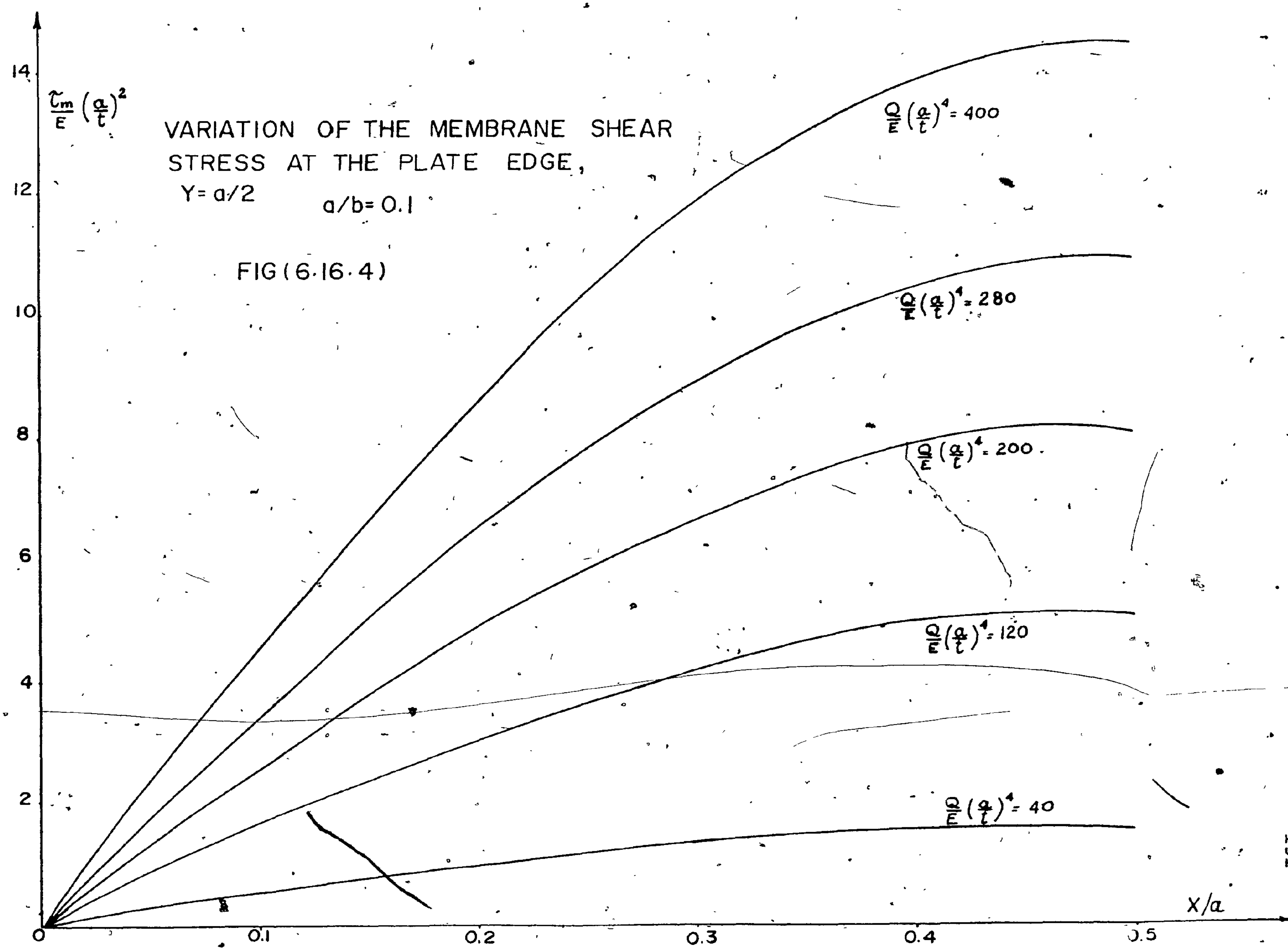
a/b = 5

FIG (6.16.2)



$\frac{\tau_m}{E}\left(\frac{a}{t}\right)^2$
VARIATION OF THE MEMBRANE SHEAR STRESS AT THE PLATE EDGE, Y = a/2
a/b = 2.5
12
10
8
6
4
2
0
0.1
0.2
0.3
0.4
0.5
X/a
$\frac{Q}{E}\left(\frac{a}{t}\right)^4 = 400$
$\frac{Q}{E}\left(\frac{a}{t}\right)^4 = 280$
$\frac{Q}{E}\left(\frac{a}{t}\right)^4 = 200$
$\frac{Q}{E}\left(\frac{a}{t}\right)^4 = 120$
$\frac{Q}{E}\left(\frac{a}{t}\right)^4 = 40$

FIG(6.16.3)



VARIATION OF THE MEMBRANE SHEAR STRESS AT THE PLATE EDGE,

Y= a/2 a/b= 0.1

FIG (6.16.4)

6.6 BENDING STRESSES

The extreme-fiber bending stresses are:

$$\sigma_{xb} = -\frac{Et}{2(1-\nu^2)}\left[\frac{\partial^2\omega}{\partial x^2} + \nu\frac{\partial^2\omega}{\partial y^2}\right] \tag{6.10}$$

$$\sigma_{yb} = -\frac{Et}{2(1-\nu^2)}\left[\frac{\partial^2\omega}{\partial y^2} + \nu\frac{\partial^2\omega}{\partial x^2}\right] \tag{6.11}$$

The values of bending stresses have been computed for ten intensities of the load, and for five values of the stiffener area, the results are plotted in Figs. (6.17) to (6.19).

For $\frac{a}{b} \to \infty$, comparison with the results obtained by Kaiser Brown and Harvey have been made. At load Q = 200 (corresponding to central deflection of 3.25 times the plate thickness) the present theory gives 16% higher stress at the plate center. The discrepancy decreases as the load decreases being, 8% at Q = 120.

The graphs of Fig. (6.17) show how the bending stress at the center of the plate given by the linear bending theory differ from those of the non-linear theory, the difference increasing with the increase in the load and the stiffener area.

A comparison of Fig. (6.17) and (6.10) shows how the ratio of the membrane stress to bending stress increases with the increasing load.

BENDING STRESSES IN THE X- DIRECTION AT THE CENTRE OF THE PLATE VS LATERAL PRESSURE

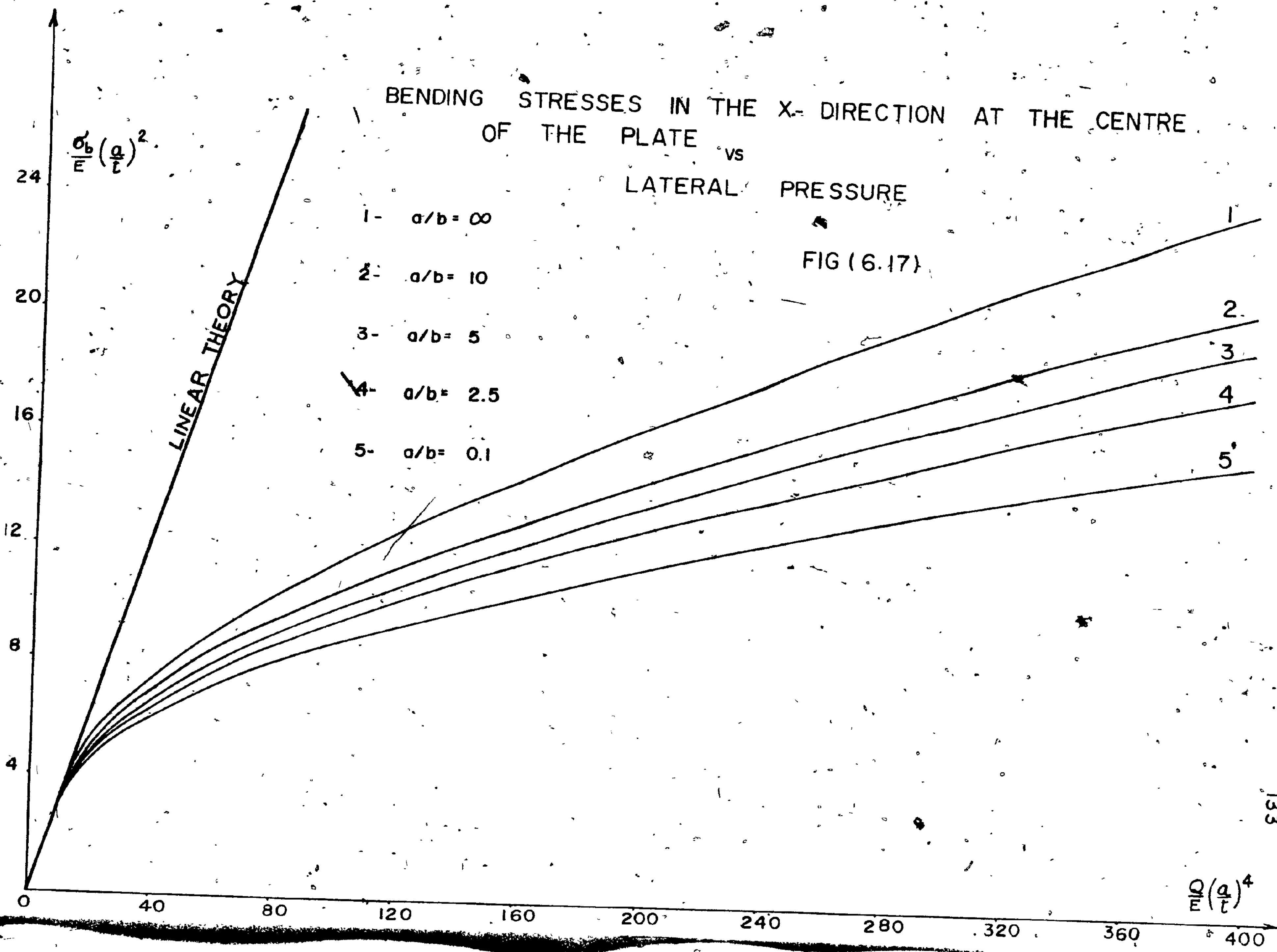


FIG (6.17)

VARIATION OF THE BENDING STRESSES ALONG SECTION Y=0

$a/b = \infty$

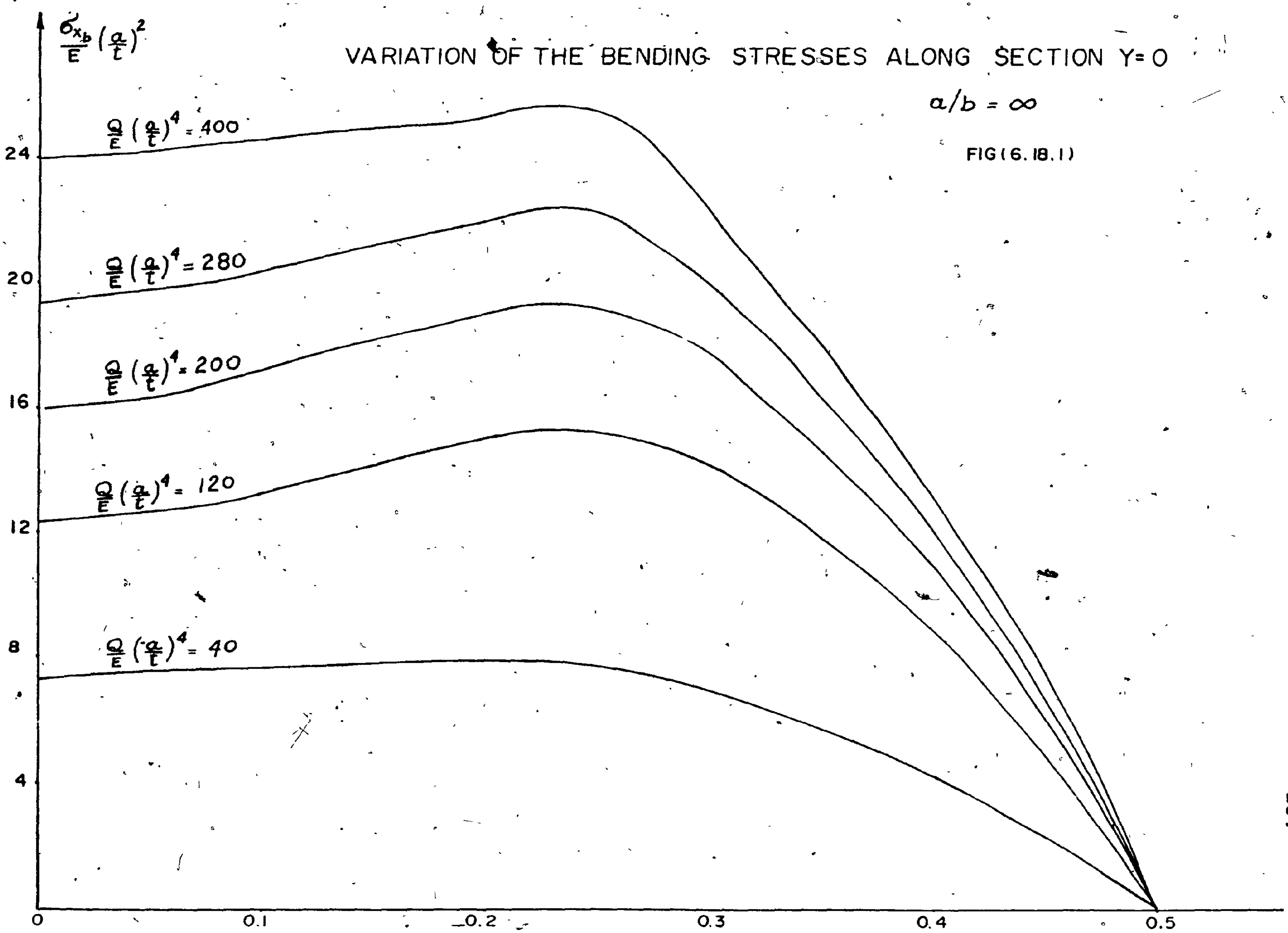
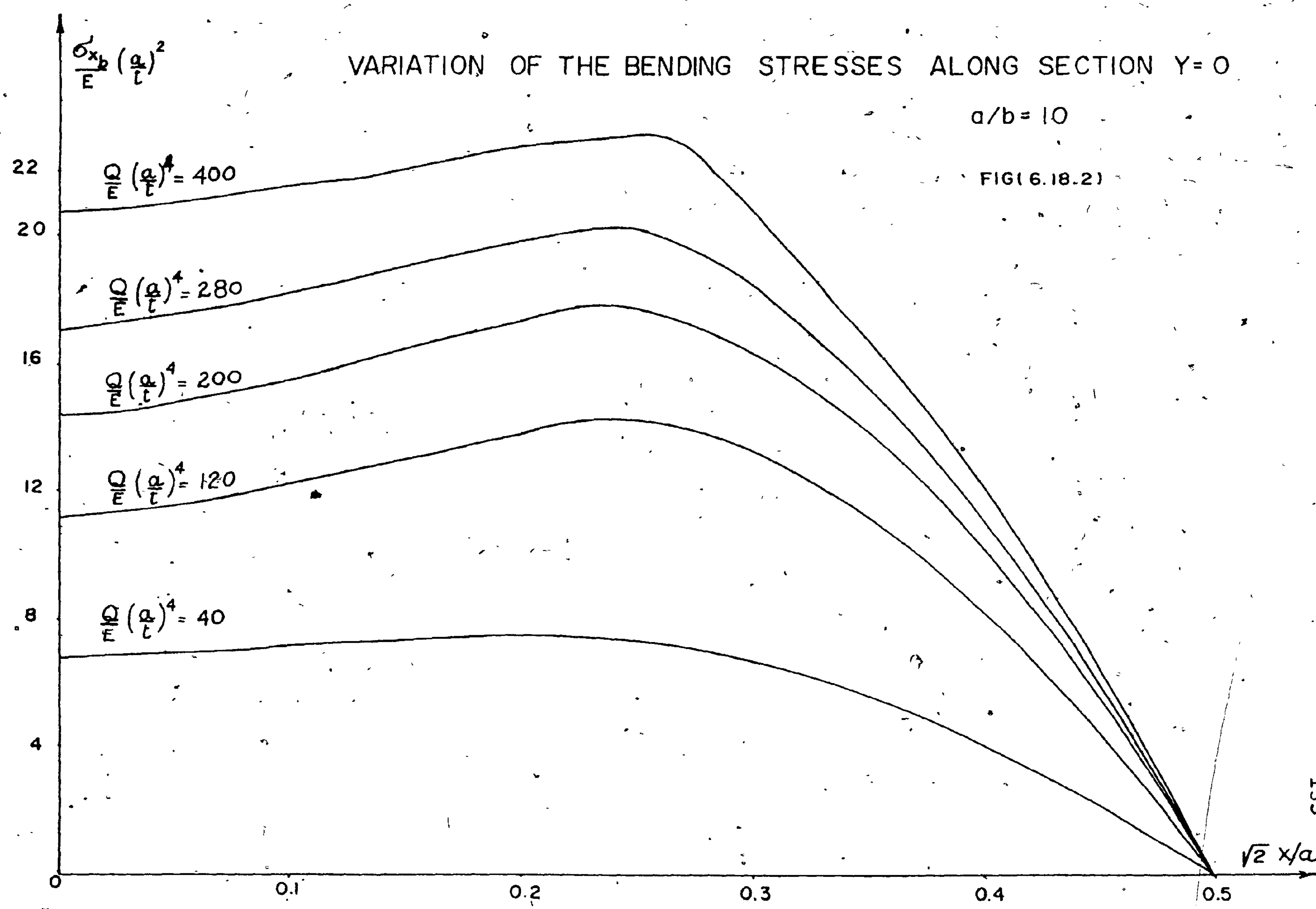


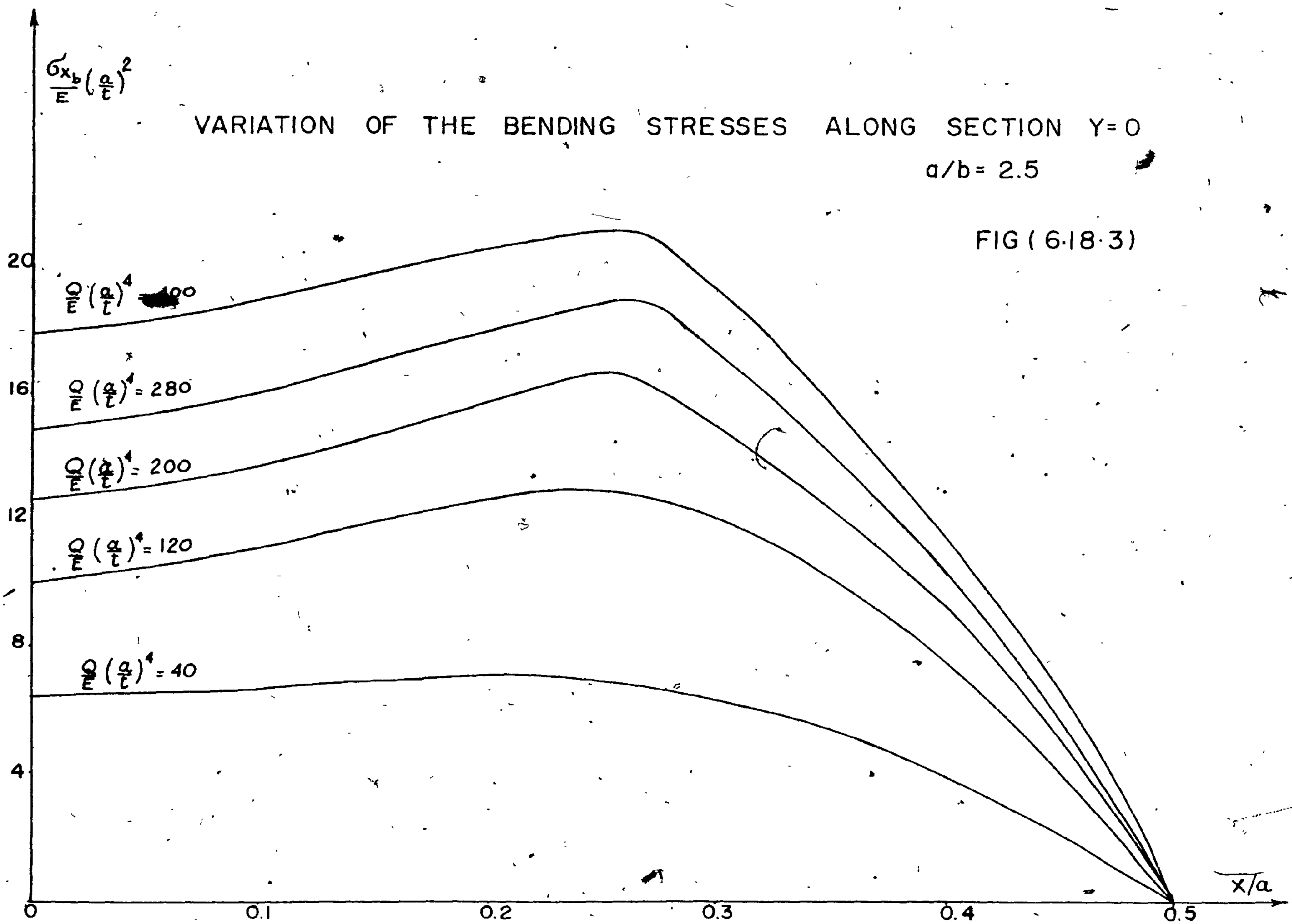
FIG (6.18.1)



VARIATION OF THE BENDING STRESSES ALONG SECTION Y = 0

a/b = 10

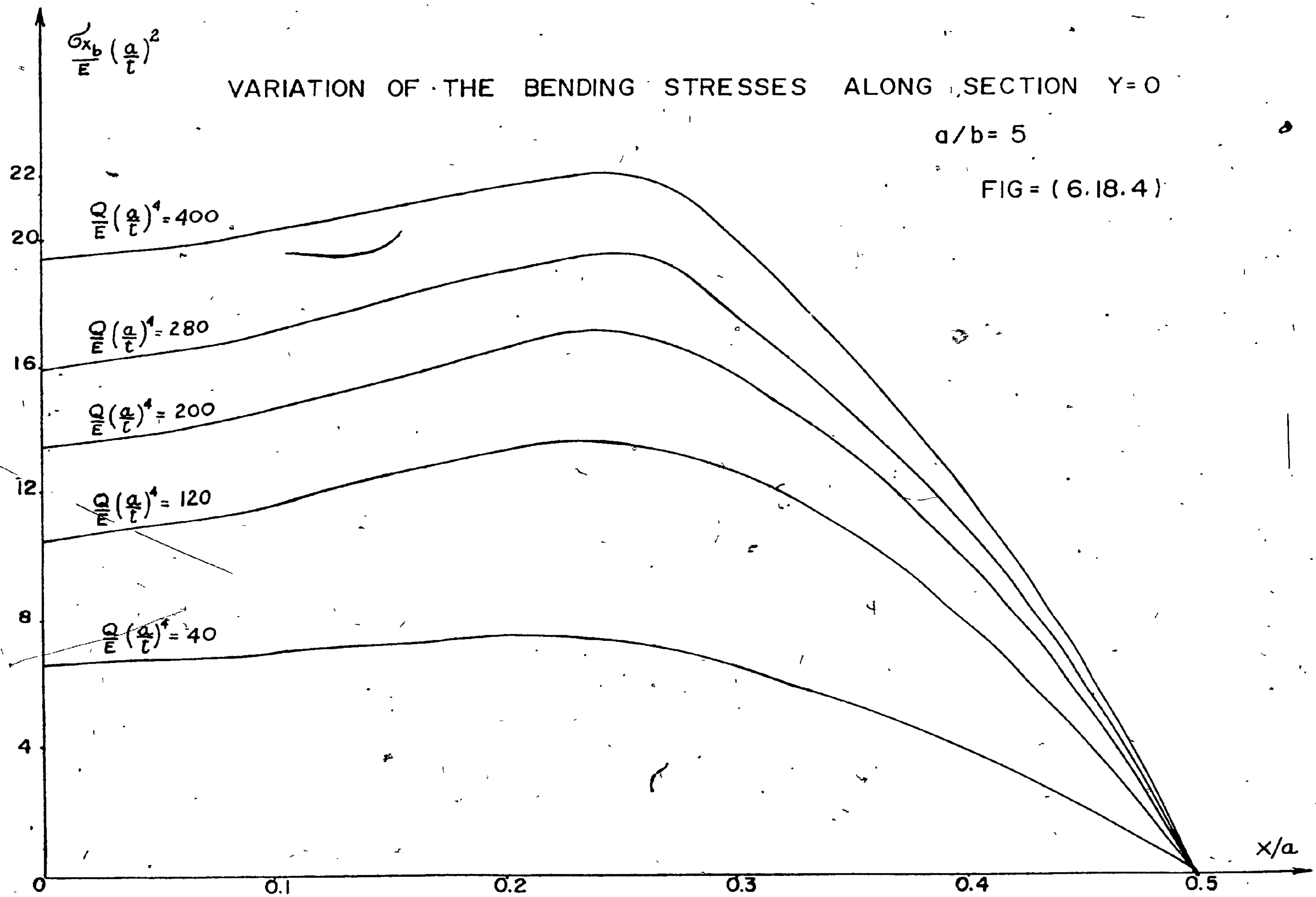
FIG (6.18.2)



VARIATION OF THE BENDING STRESSES ALONG SECTION Y=0

a/b= 2.5

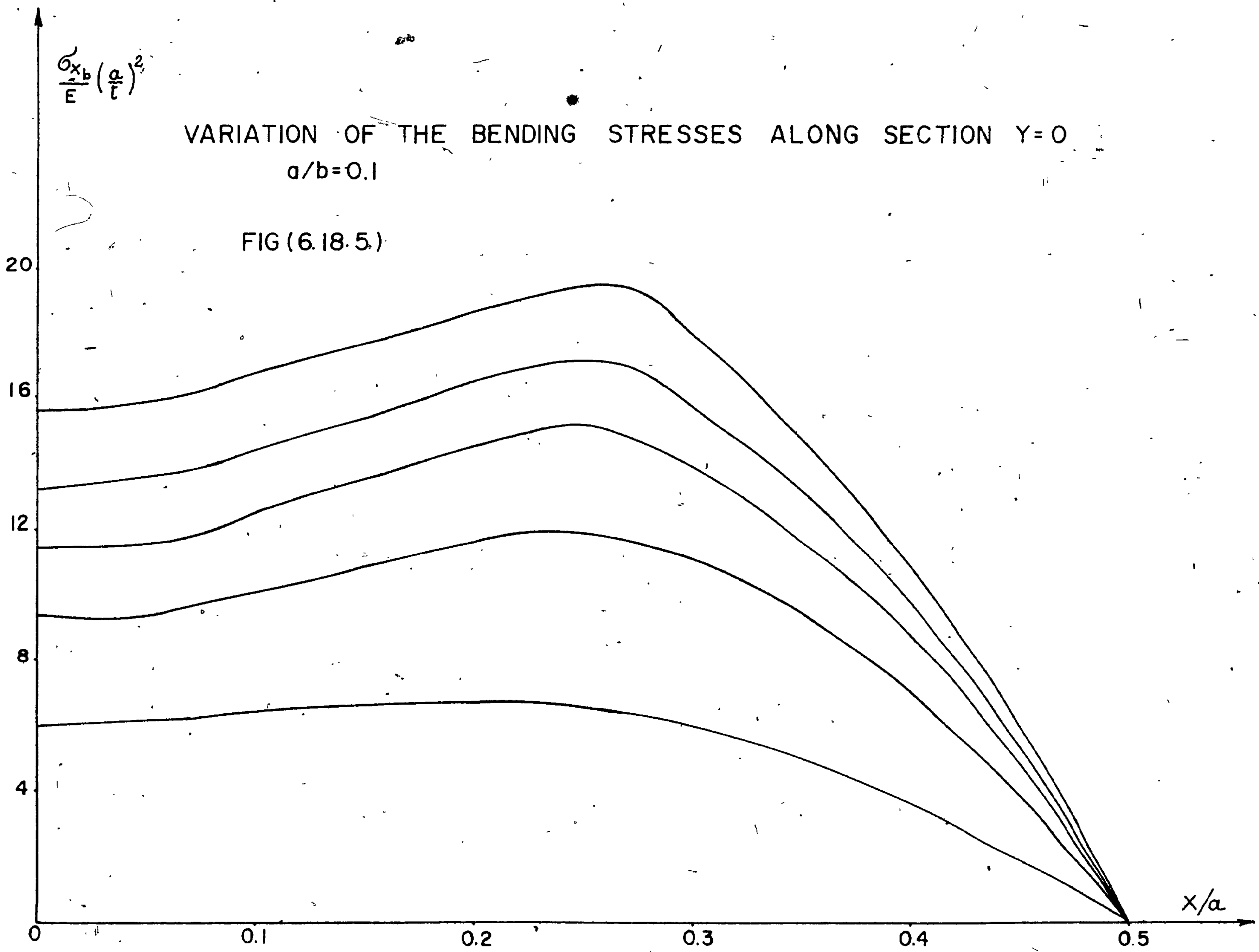
FIG (6.18.3)



VARIATION OF THE BENDING STRESSES ALONG SECTION Y = 0

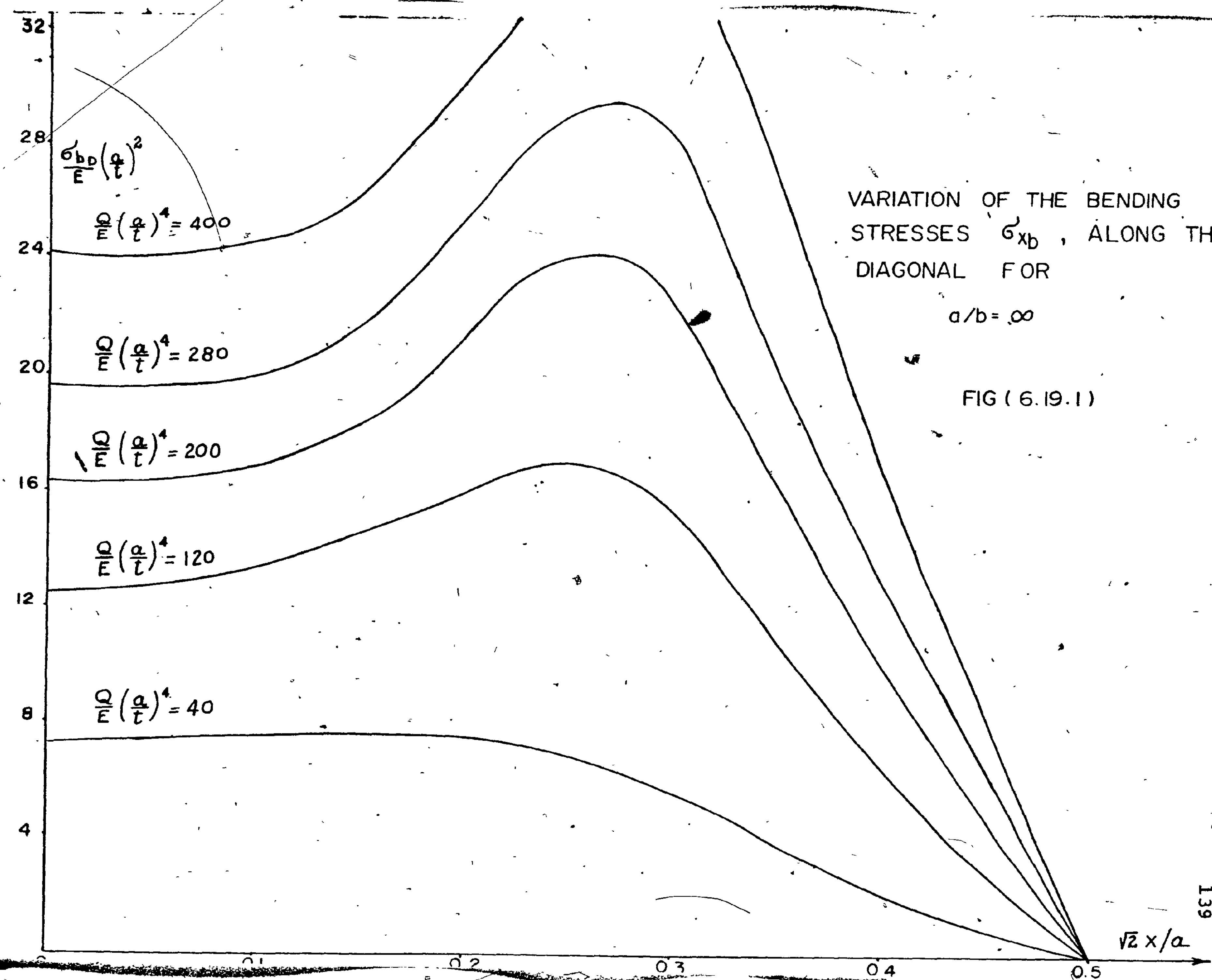
a/b = 5

FIG = (6.18.4)



VARIATION OF THE BENDING STRESSES ALONG SECTION Y=0
a/b=0.1

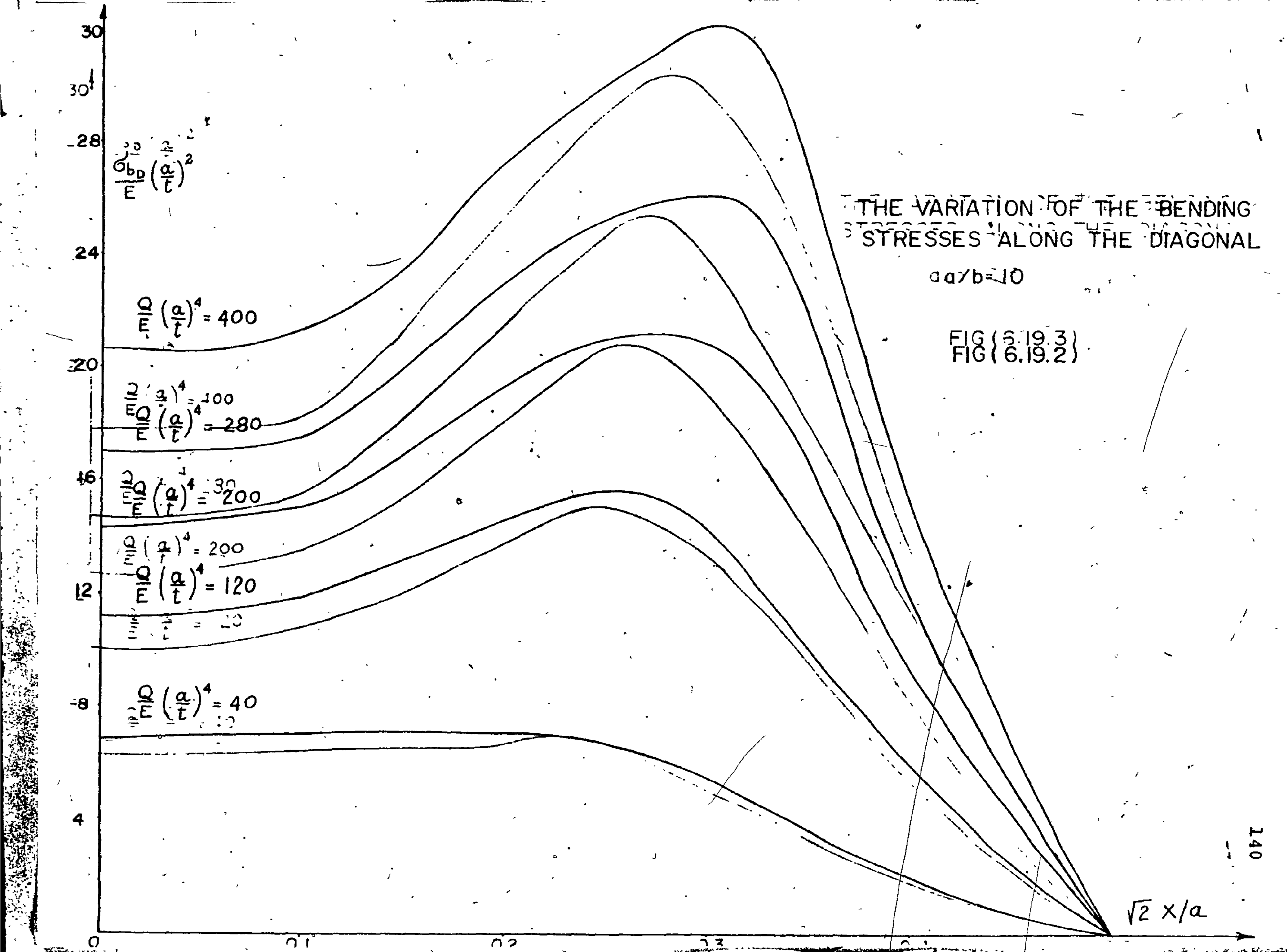
FIG (6.18.5)



VARIATION OF THE BENDING STRESSES σ_{xb}, ALONG THE DIAGONAL FOR

$a/b = \infty$

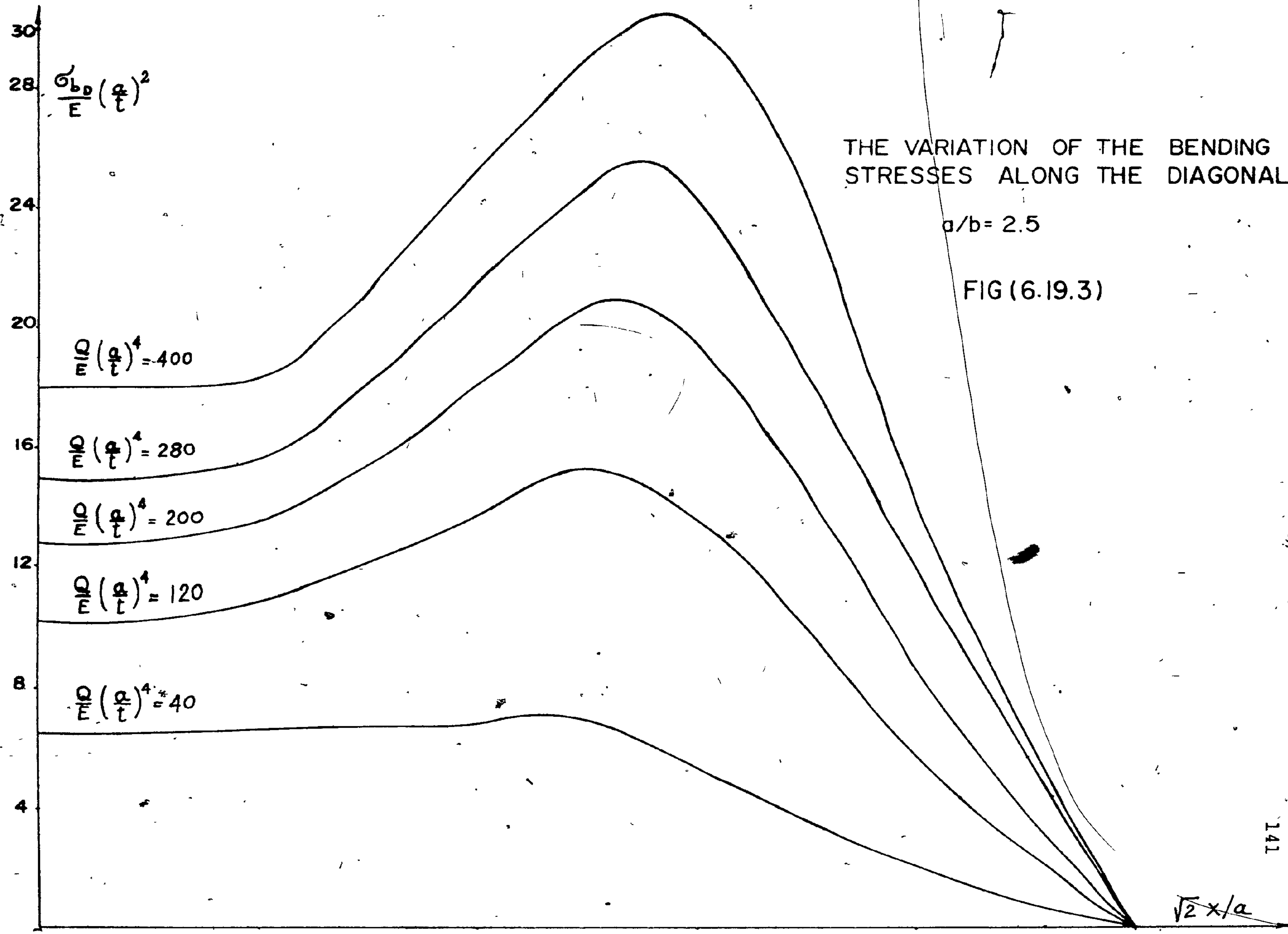
FIG (6.19.1)



THE VARIATION OF THE BENDING STRESSES ALONG THE DIAGONAL

$a/b = 1.0$

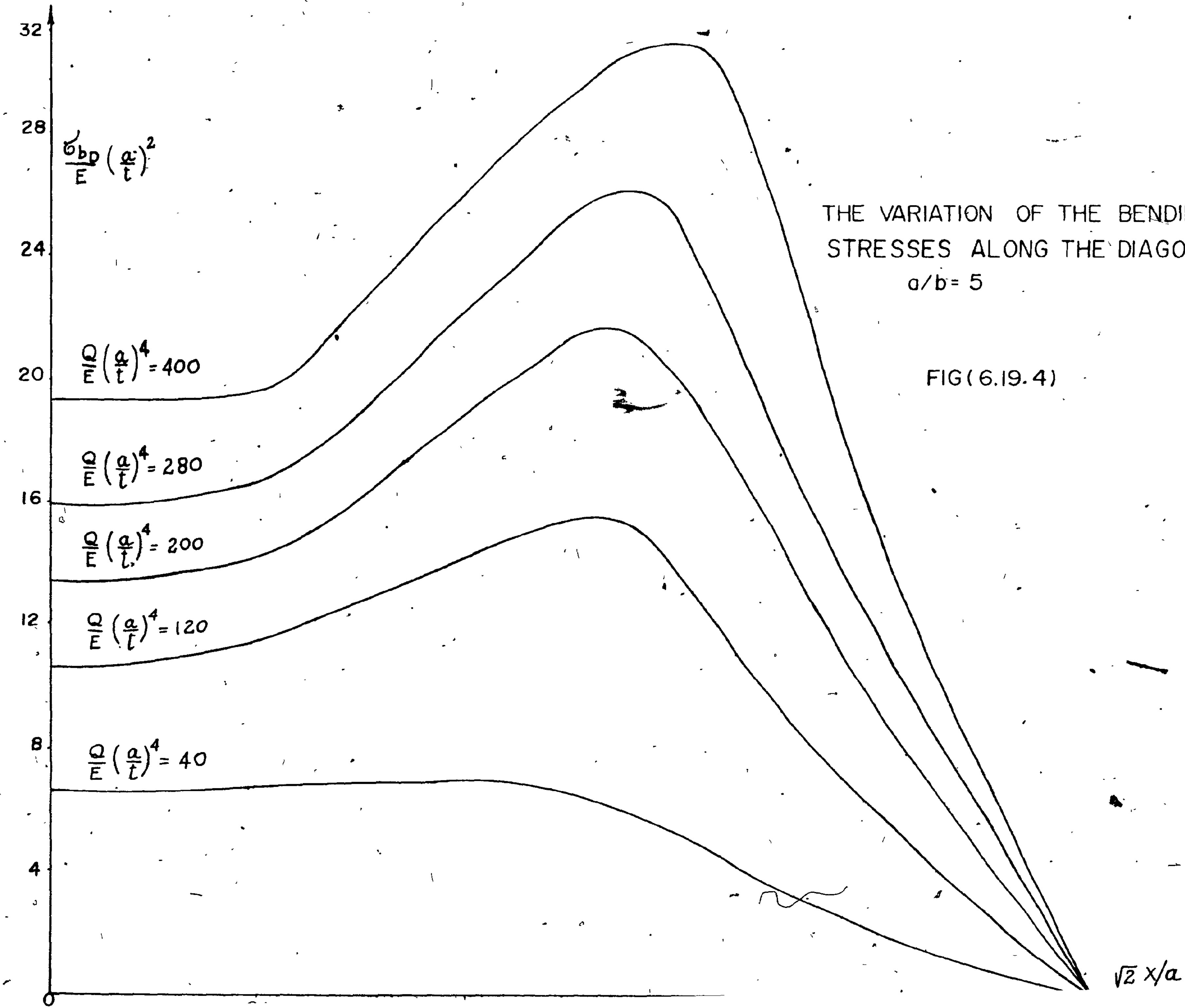
FIG (6.19.3)
FIG (6.19.2)



THE VARIATION OF THE BENDING STRESSES ALONG THE DIAGONAL

$a/b = 2.5$

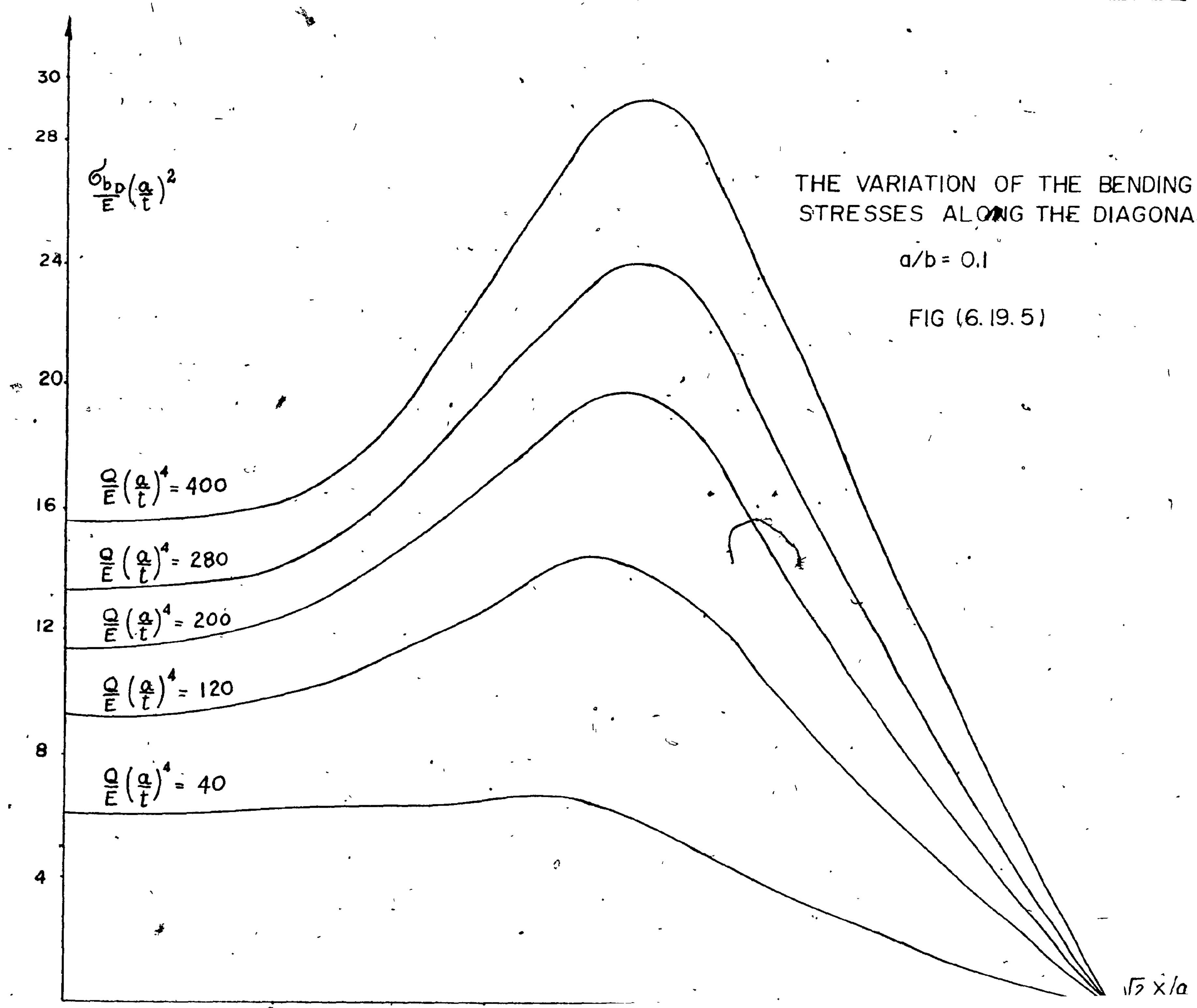
FIG (6.19.3)



THE VARIATION OF THE BENDING STRESSES ALONG THE DIAGONAL

$a/b = 5$

FIG (6.19.4)



THE VARIATION OF THE BENDING STRESSES ALONG THE DIAGONAL

$a/b = 0.1$

FIG (6.19.5)

The bending stress, σ_{xb} along the center line of the plate is plotted against the load in Fig. (6.18).

Curves of the bending stresses have been drawn corresponding to five values of the transverse load Q. From linear theory, the maximum bending stresses are at the center. Figure (6.18) shows that there is a local minimum at the center, and the maximum which occurs between the center and the edge, moves towards the edges as Q increases.

In Fig. (6.19), the variation of the bending stress in the x-direction along the plate diagonal is plotted. The change in stress pattern with increasing Q is clearly seen. The maximum stress would occur at the center for low Q, but peaks develop on the diagonal away from the center for large values of Q. The maximum bending stress in the x-direction is on the diagonal.

6.7 SUMMARY AND CONCLUSIONS

The variational technique, which is programmed on the digital computer, is employed to solve the problem of large deflection of square isotropic plates stiffened along the boundaries. The governing non-linear differential equations were reduced to a system of non-linear algebraic equations.

In the analysis shown in the present work, load levels have been taken of sufficient magnitude to cause deflections of about 4.5 times the plate thickness, these load levels are higher than those examined by previous investigators for the solution of unstiffened plate problems. Among all the methods used, the variational technique takes the least time on the computer.

Numerical results showing the effect of the variation of the cross-sectional area of the boundary stiffeners on the plate deformability are given. To verify the present method, the deflection and stresses for a non-stiffened plate were compared with the results of other authors, with close agreement.

This study has been confined to a square plate with a width-to-thickness area of 200, for which the following observations are made:

1) The load to create a deflection of a little more than three times the plate thickness in a plate with large stiffeners is double that for the unstiffened plate, while with stiffeners with cross-sectional area equal to .2 of the cross-sectional area of the plate would require a load of 1.4 times that for unstiffened plate for the same deflection.

2) In an unstiffened plate, for a similar deflection, i.e., a little over 3t, the maximum compressive membrane stress at the boundary is almost double the tension stress at the center, whereas with a stiffener with b = .2a, the ratio is approximately 1.2 with a reduction of 7% in the absolute value of the stress at the

center. With large stiffener area, the boundary normal membrane stress approaches zero.

3) The lines of zero normal membrane stresses (neutral line) given by this analysis are straight and parallel to the edge, and the location of these lines is essentially independent of the loading. As the stiffener area increases, the lines move towards the boundaries until with large stiffener area, they lie adjacent to the stiffeners.

4) The maximum membrane shear stress lies on the intersection of the neutral lines. Its magnitude is approximately .7 of the membrane stress at the center for the above case with $b = .2t$ rising to 1.2 for large stiffener area.

5) For a plate with large stiffener area ($b > a$), the corners are not displaced although the edges of the plate still translate towards the center.

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REFERENCES

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APPENDIX A

INTEGRATION TABLE

APPENDIX A

INTEGRATION TABLE

$$\int_0^{a/2} \cos\frac{\pi x}{a}\,dx = \int_0^{a/2} \sin\frac{\pi x}{a}\,dx = \frac{a}{\pi}$$

$$\int_0^{a/2} \sin^2 n\frac{\pi x}{a}\,dx = \int_0^{a/2} \cos^2 n\frac{\pi x}{a}\,dx = \frac{a}{4}$$

where $n=1,2,3,\ldots$

$$\int_0^{a/2} \cos^3 \frac{\pi x}{a}\,dx = \frac{2}{3}\,\frac{a}{\pi}$$

$$\int_0^{a/2} \cos^4 \frac{\pi x}{a}\,dx = \int_0^{a/2} \sin^4 \frac{\pi x}{a}\,dx = \frac{3}{16}\,a$$

$$\int_0^{a/2} \cos^5 \frac{\pi x}{a}\,dx = \frac{a}{\pi}\,\frac{8}{15}$$

$$\int_0^{\pi/2} \cos^6\theta\,d\theta = \frac{5}{32}\,\pi$$

$$\int_0^{\pi/2} \cos^8\theta\,d\theta = \frac{35}{256}\,\pi$$

$$\int_0^{a/2} \cos \frac{2n\pi x}{a}\,dx = 0$$

where $n=1,2,\ldots$

$$\int_0^{a/2} \cos\frac{3\pi x}{a}\,dx = -\frac{a}{3\pi}$$

$$\int_0^{a/2} \cos^3 \frac{3\pi x}{a}\, dx = -\frac{2}{9}\,\frac{a}{\pi}$$

$$\int_0^{a/2} \sin \frac{3\pi x}{a}\, dx = \frac{a}{3\pi}$$

$$\int_0^{a/2} \sin^4 \frac{3\pi x}{a}\, dx = \frac{3}{16}\, a$$

$$\int_0^{a/2} \left(\frac{\pi x}{a}\right) \sin \frac{\pi x}{a}\, dx = \frac{a}{\pi}$$

$$\int_0^{a/2} \left(\frac{\pi x}{a}\right) \sin \frac{2\pi x}{a}\, dx = \frac{a}{4}$$

$$\int_0^{a/2} \left(\frac{\pi x}{a}\right) \sin \frac{3\pi x}{a}\, dx = -\frac{a}{9\pi}$$

$$\int_0^{a/2} \left(\frac{\pi x}{a}\right) \sin \frac{4\pi x}{a}\, dx = -\frac{a}{8}$$

$$\int_0^{a/2} \left(\frac{\pi x}{a}\right) \sin \frac{6\pi x}{a}\, dx = \frac{a}{12}$$

$$\int_0^{a/2} \cos \frac{\pi x}{a} \cos \frac{2\pi x}{a}\, dx = \frac{1}{3}\,\frac{a}{\pi}$$

$$\int_0^{a/2} \cos \frac{\pi x}{a} \cos \frac{3\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \cos \frac{2\pi x}{a} \cos \frac{3\pi x}{a}\, dx = \frac{3}{5}\,\frac{a}{\pi}$$

$$\int_0^{a/2} \cos\frac{2\pi x}{a} \cos\frac{4\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \cos\frac{\pi x}{a} \cos\frac{4\pi x}{a}\, dx = -\frac{1}{15}\frac{a}{\pi}$$

$$\int_0^{a/2} \cos\frac{3\pi x}{a} \cos\frac{4\pi x}{a}\, dx = \frac{3}{7}\frac{a}{\pi}$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \sin\frac{6\pi x}{a}\, dx = \frac{a}{\pi}\left(\frac{6}{35}\right)$$

$$\int_0^{a/2} \sin\frac{2\pi x}{a} \sin\frac{3\pi x}{a}\, dx = \frac{2a}{5\pi}$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \sin\frac{2\pi x}{a}\, dx = \frac{2a}{3\pi}$$

$$\int_0^{a/2} \sin\frac{2\pi x}{a} \sin\frac{6\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \sin\frac{3\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \sin\frac{3\pi x}{a} \sin\frac{4\pi x}{a} = \frac{4}{7}\frac{a}{\pi}$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \sin\frac{4\pi x}{a}\, dx = -\frac{4}{15}\frac{a}{\pi}$$

$$\int_0^{a/2} \sin\frac{2n\pi x}{a} \sin\frac{2m\pi x}{a}\, dx = 0$$

where $m \neq n, m=1,2,\ldots, n=1,2,3,\ldots$

$$\int_0^{a/2} \sin^2\frac{\pi x}{a}\cos^2\frac{\pi x}{a}\,dx = \frac{a}{16}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a}\cos^2\frac{3\pi x}{a}\,dx = \frac{a}{8}$$

$$\int_0^{a/2} \sin^2\frac{\pi x}{a}\sin^2\frac{3\pi x}{a}\,dx = \frac{a}{8}$$

$$\int_0^{a/2} \cos^2\frac{3\pi x}{a}\sin^2\frac{\pi x}{a}\,dx = \frac{a}{8}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a}\sin^2\frac{3\pi x}{a}\,dx = \frac{a}{8}$$

$$\int_0^{a/2} \sin^2\frac{3\pi x}{a}\cos^2\frac{3\pi x}{a}\,dx = \frac{a}{16}$$

$$\int_0^{a/2} \sin^2\frac{\pi x}{a}\cos\frac{\pi x}{a}\,dx = \frac{a}{3\pi}$$

$$\int_0^{a/2} \sin^2\frac{\pi x}{a}\cos\frac{3\pi x}{a}\,dx = -\frac{7}{15}\frac{a}{\pi}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a}\cos\frac{3\pi x}{a}\,dx = \frac{2}{15}\frac{a}{\pi}$$

$$\int_0^{a/2} \cos\frac{\pi x}{a}\sin^2\frac{3\pi x}{a}\,dx = \frac{17a}{35\pi}$$

$$\int_0^{a/2} \cos\frac{3\pi x}{a}\sin^2\frac{3\pi x}{a}\,dx = -\frac{a}{9\pi}$$

$$\int_0^{a/2} \cos\frac{\pi x}{a} \cos^2\frac{3\pi x}{a}\, dy = \frac{18}{35}\,\frac{a}{\pi}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a} \cos\frac{3\pi x}{a}\, dx = \frac{a}{16}$$

$$\int_0^{a/2} \cos\frac{\pi x}{a} \cos^3\frac{3\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \cos\frac{2\pi x}{a} \cos^2\frac{\pi x}{a}\, dx = \frac{a}{8}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a} \cos\frac{4\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \sin\frac{3\pi x}{a} \sin^3\frac{\pi x}{a}\, dx = -\frac{a}{16}$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \sin^3\frac{3\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \cos\frac{2\pi x}{a} \sin^2\frac{\pi x}{a}\, dx = -\frac{a}{8}$$

$$\int_0^{a/2} \cos\frac{4\pi x}{a} \sin^2\frac{\pi x}{a}\, dx = 0$$

$$\int_0^{a/2} \sin\frac{\pi x}{a} \cos\frac{\pi x}{a} \sin\frac{3\pi x}{a}\, dx = \frac{a}{5\pi}$$

$$\int_0^{a/2} \left(\sin^2\frac{\pi x}{a} \cos\frac{\pi x}{a} \cos\frac{3\pi x}{a}\right) dx = -\frac{a}{16}$$

$$\int_0^{a/2} \cos^2\frac{\pi x}{a}\,\sin\frac{\pi x}{a}\,\sin\frac{3\pi x}{a}\,dx = \frac{a}{16}$$

$$\int_0^{a/2} \cos^2\frac{3\pi x}{a}\,\sin\frac{\pi x}{a}\,\sin\frac{3\pi x}{a}\,dx = 0$$

$$\int_0^{a/2} \sin^2\frac{3\pi x}{a}\,\cos\frac{\pi x}{a}\,\cos\frac{3\pi x}{a}\,dx = 0$$

APPENDIX B

COMPUTER ANALYSIS

APPENDIX B

COMPUTER ANALYSIS

B.1 COMPUTER NOTATION

Variable	Description
A	Square matrix of size 28×28 defined by Equation (5.36)
ANIV	Output matrix of Dimension 28×28 containing the inverse of A
WKAREA	Work area of dimension 868
B	Rectangular matrix of size 28×6 defined by Equation (5.36)
C	Result matrix of size 28×6 defined by Equation (5.37)
D11	Row matrix of size 1×28 defined by Equation (5.38)
D12	Row matrix of size 1×28 defined by Equation (5.39)
D13	Row matrix of size 1×28 defined by Equation (5.39)
D22	Row matrix of size 1×28 defined by Equation (5.41)
D23	Row matrix of size 1×28 defined by Equation (5.41)
D33	Row matrix of size 1×28 defined by Equation (5.43)
E1	Result matrix of size (6×1) defined by Equation (5.39)
E2	Result matrix of size (6×1) defined by Equation (5.39)
E3	Result matrix of size (6×1) defined by Equation (5.39)

E4	Result matrix of size (6×1) defined by Equation (5.41)
E5	Result matrix of size (6×1) defined by Equation (5.41)
E6	Result matrix of size (6×1) defined by Equation (5.43)
AK	The load factor takes the value of .1, .2, .3, ..., 1
F	The name of the function called by 2SYSTM to furnish the values of the functions which define the system of equations being solved.
MU	ν (Poisson's ratio has the value of .316)
LINV2F	Subroutine for matrix inversion, full storage mode, high accuracy solution
UMULFF	Subroutine for matrix multiplication
X	A vector of length 3. As input, is the initial guess to the root. As output, it is the computed solution ($\delta_1, \delta_2, \delta_3$)
GS	A vector of length 28 containing the computed solution (C_1, C_2, C_3, K , ..., λ_{13})
AB	The ratio a/b
AT	The ratio a/t
SEGT	Membrane tension stresses at the plate centre
SEGC	Membrane compressive stresses at Plate mid-edges
ESEG	Amount of error defined by Equation (6).
SEGTN	Membrane tension stresses at the plate centre multiplied by $(a/t)^2$
SEGCN	Membrane compressive stresses at plate mid-edges multiplied by $(a/t)^2$

XLOC	Locus of the neutral line
SEGB	Bending stresses at the plate centre
DEFS	Vertical deflection across the section y=o
DEFDN	Vertical deflection across the diagonal section
SEGBS1	Bending stresses in the x-direction across section y=o
SEGBS2	Bending stresses in the y-direction across section y=o
TAOM	Membrane shear stress across the plate edge
TAOMD	Membrane shear stress across the plate diagonal
SEGBD	Bending stresses across the diagonal
TAOBD	Extreme fiber shearing stress across the diagonal
ZSYSTM	Subroutine for determination of the roots of a system of N simultaneous nonlinear equations

B2 COMPUTER PROGRAM - METHOD 1

```
      PROGRAM ANEES (INPUT,OUTPUT)
      DIMENSION A(28,28),AINV(28,28),WKAREA(868),B(28,6),C(28,6),
     1D11(1,28),E1(6),D12(1,28),E2(6),D13(1,28),E3(6),
     2D22(1,28),E4(6),D23(1,28),E5(6),D33(1,28),E6(6)
     3,X(3),WA(14),FS(6,1),GS(28,1)
      COMMON AK,E1,E2,E3,E4,E5,E6
      EXTERNAL    F
      REAL MU
      AB=.0947116
       AT=200.
      MU=.316
      PI=3.1415927
      N=28
      IA=28
      IDGT=7
      DO 20 I=1,28
      DO 10 J=1,28
      READ*,A(I,J)
 10    CONTINUE
 20   CONTINUE
      A(1,1)=A(1,1)+(1.-MU**2)*(PI**2)/AB
      PRINT 101,((A(I,J),J=1,10),I=1,28)
      PRINT 101,((A(I,J),J=11,20),I=1,28)
      PRINT 102,((A(I,J),J=21,28),I=1,28)
 101  FORMAT (1H1/10(3X,F10.6)//)
 102  FORMAT (1H1/8(3X,F10.6)//)
      CALL LINV2F (A,N,IA,AINV,IDGT,WKAREA,IER)
      PRINT 101,((AINV(I,J),J=1,10),I=1,28)
      PRINT 101,((AINV(I,J),J=11,20),I=1,28)
      PRINT 102,((AINV(I,J),J=21,28),I=1,28)
      PRINT*,IDGT,IER
       L=28
       M=28
       N=6
        IA=28
       IB=28
       IC=28
      DO 30 I=1,28
      DO 40 J=1,6
      READ*,B(I,J)
 40   CONTINUE
 30   CONTINUE
      PRINT 100,((B(I,J),J=1,6),I=1,28)
 100  FORMAT(1H1/6(5X,E15.6)//)
      CALL VMULFF (AINV,B,L,M,N,IA,IB,C,IC,IER)
      PRINT 100,((C(I,J),J=1,6),I=1,28)
       PRINT*,IER
      L=1
      M=28
      N=6
       IA=1
      IB=28
       IC=1
      DO 50 J=1,28
      READ*,D11(1,J)
 50   CONTINUE
      PRINT 100,(D11(1,J),J=1,28)
```

```
      CALL VMULFF (D11,C,L,M,N,IA,IB,E1,IC,IER)
      PRINT 100,(E1(J),J=1,6)
      PRINT*,IER
      DO 60 J=1,28
      READ*,D12(1,J)
60    CONTINUE
      PRINT 100,(D12(1,J),J=1,28)
      CALL VMULFF (D12,C,L,M,N,IA,IB,E2,IC,IER)
      PRINT 100,(E2(J),J=1,6)
      PRINT*,IER
      DO 70 J=1,28
      READ*,D13(1,J)
70    CONTINUE
      PRINT 100,(D13(1,J),J=1,28)
      CALL VMULFF (D13,C,L,M,N,IA,IB,E3,IC,IER)
      PRINT 100,(E3(J),J=1,6)
      PRINT*,IER
      DO 80 J=1,28
      READ*,D22(1,J)
80    CONTINUE
      PRINT 100,(D22(1,J),J=1,28)
      CALL VMULFF (D22,C,L,M,N,IA,IB,E4,IC,IER)
      PRINT 100,(E4(J),J=1,6)
      PRINT*,IER
      DO 90 J=1,28
      READ*,D23(1,J)
90    CONTINUE
      PRINT 100,(D23(1,J),J=1,28)
      CALL VMULFF (D23,C,L,M,N,IA,IB,E5,IC,IER)
      PRINT 100,(E5(J),J=1,6)
      PRINT*,IER
      DO 110 J=1,28
      READ*,D33(1,J)
110   CONTINUE
      PRINT 100,(D33(1,J),J=1,28)
      CALL VMULFF (D33,C,L,M,N,IA,IB,E6,IC,IER)
      PRINT 100,(E6(J),J=1,6)
      PRINT*,IER
      DO 120 J=1,10
      AK=J/10.0
      X(1)=.02
      X(2)=-.001
      X(3)=.001
      EPS=.00000001
      NSIG=5
      N=3
      ITMAX=50
      CALL ZSYSTM (F,EPS,NSIG,N,X,ITMAX,WA,PAR,IER)
      DO 130 K=1,3
      D=F(X,K,0.)
      PRINT*,D
      PRINT 1001
1001  FORMAT (1H1)
130   CONTINUE
      PRINT*,(X(I),I=1,3)
      PRINT*,IER,ITMAX
      PRINT*,AK
```

```
      DN=X(1)+2*X(2)+X(3)
      PRINT*,DN
      DNN=AT*DN
      PRINT*,DNN
      FS(1,1)=(X(1))**2
      FS(2,1)=(X(2))**2
      FS(3,1)=(X(3))**2
      FS(4,1)=(X(1))*(X(2))
      FS(5,1)=(X(1))*(X(3))
      FS(6,1)=(X(2))*(X(3))
      PRINT 100,(FS(K,1),K=1,6)
      L=28
      M=6
      N=1
      IA=28
      IB=6
      IC=28
      CALL VMULFF (C,FS,L,M,N,IA,IB,GS,IC,IER)
      PRINT 103,(GS(I,1),I=1,28)
  103 FORMAT (1H1/(5X,E15.6)//)
      PRINT*,IER
      SEGT=PI*GS(1,1)+(PI*GS(3,1))/(1-MU)
      SEGC=PI*GS(1,1)
      ESEG=((1-MU**2)*(GS(1,1))/(AB)+((1-MU**2)*(GS(1,1))/2.0)
     1+((1.316)*GS(3,1)/PI))/((1.+MU)*GS(3,1)/PI)
      SEGTN=SEGT*AT*AT
      SEGCN=SEGC*AT*AT
      PRINT*,SEGT,SEGC,ESEG
      PRINT*,SEGTN,SEGCN
      XLOC=ACOS(-(1.-MU**2)*GS(1,1)/(GS(3,1)*(1.+MU)))/PI
      SEGB=PI*PI*AT*(X(1)+10.*X(2)+9.*X(3))/(2.*(1.-MU))
      DO 140 K=1,6
      ARG=(K-1.)/10.
      XY=PI*ARG
      DEFS=COS(XY)*(X(1)+X(2))+COS(3.*XY)*(X(2)+X(3))
      DEFDN=AT*(COS(XY)*COS(XY)*X(1)+2.*COS(XY)*COS(3.*XY)*X(2)
     1+COS(3.*XY)*COS(3.*XY)*X(3))
      DEFSN=AT*DEFS
      SEGBS1=PI*PI*AT*(COS(XY)*((1.+MU)*X(1)+X(2)+9.*MU*X(2))
     1+COS(3.*XY)*(9.*(1.+MU)*X(3)+9.*X(2)+MU*X(2)))/(2.*(1.-MU**2))
      SEGBS2=PI*PI*AT*(COS(XY)*((1.+MU)*X(1)+9.*X(2)+MU*X(2))
     1+COS(3.*XY)*(X(2)+9.*MU*X(2)+9.*(1.+MU)*X(3)))/(2.*(1.-MU**2))
      TAOM=AT*AT*(ARG*PI*PI*GS(1,1)+(PI*SIN(XY)*GS(3,1)/(1.-MU)))
      TAOMD=AT*AT*SIN(XY)*(ARG*PI*PI*GS(1,1)+(PI*SIN(XY)*GS(3,1)/
     1(1.-MU)))
      SEGBD=PI*PI*AT*(COS(XY)*COS(XY)*(1.+MU)*X(1)+COS(XY)*COS(3.*XY)*
     1(1.+MU)*10.0*X(2)+COS(3.*XY)*COS(3.*XY)*9.*(1.+MU)*X(3))/(2.*
     2(1.-MU**2))
      TAOBD=-PI*PI*AT*(SIN(XY)*SIN(XY)*X(1)+6.*X(2)*SIN(XY)*SIN(3.*XY)
     1+9.*X(3)*SIN(3.*XY)*SIN(3.*XY))/(2.*(1.+MU))
      PRINT*,ARG,SEGBS1,SEGBS2,SEGBD
      PRINT*,ARG,TAOM,TAOMD
      PRINT*,ARG,DEFSN,DEFDN
      PRINT*,ARG,TAOBD
  140 CONTINUE
      PRINT*,XLOC
      PRINT*,SEGB
  120 CONTINUE
      STOP
      END
```

```
      FUNCTION F(X,K,PAR)
      DIMENSION X(3)
      DIMENSION E1(6),E2(6),E3(6),E4(6),E5(6),E6(6)
      COMMON AK,E1,E2,E3,E4,E5,E6
      A=X(1)
      B=X(2)
      C=X(3)
      IF(K.EQ.1) F=(E1(1)+.52052)*A**3+(E2(2)+27.0676)*B**3+(E3(3))*C**3
     1+(E1(4)+E2(1)+3.12318)*B*A**2+(E1(2)+E2(4)+17.698)*A*B**2
     2+(E1(5)+E3(1)-4.6848)*C*A**2+(E1(3)+E3(5)+18.739)*A*C**2
     3+(E2(6)+E3(2))*C*B**2+(E2(3)+E3(6))*B*C**2
     4+(E1(6)+E2(5)+E3(4)-12.49272)*A*B*C+(.000101468)*A+.0000091204*AK
      IF(K.EQ.2) F=(E2(1)+1.04106)*A**3+(E4(2)+180.1032)*B**3
     1+(E5(3))*C**3+(E2(4)+E4(1)+17.698)*B*A**2+(E2(2)+E4(4)+81.2028)*A
     2*B**2+(E2(5)+E5(1)-6.24636)*C*A**2+(E2(3)+E5(5))*A*C**2
     3+(E4(6)+E5(2)+28.1085)*C*B**2+(E4(3)+E5(6)+93.6954)*B*C**2
     4+(E2(6)+E4(5)+E5(4))*A*B*C+.0050734*B+.0000060802*AK
      IF(K.EQ.3) F=(E3(1)-1.5616)*A**3+(E5(2)+9.3695)*B**3+
     1(E6(3)+42.1628)*C**3+(E3(4)+E5(1)-6.24636)*B*A**2
     2+(E3(2)+E5(4))*A*B**2+(E3(5)+E6(1)+18.739)*C*A**2
     3+(E3(3)+E6(5))*A*C**2+(E5(6)+E6(2)+93.6954)*C*B**2
     4+(E5(3)+E6(6))*B*C**2+(E3(6)+E5(5)+E6(4))*A*B*C
     5+.0082188*C-.0000010134*AK
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

```
OINTS
F

ES      SN  TYPE        RELOCATION
A           REAL                               0   AK      REAL              / /
B           REAL                             322   C       REAL
E1          REAL      ARRAY    / /             7   E2      REAL     ARRAY    / /
E3          REAL      ARRAY    / /            23   E4      REAL     ARRAY    / /
E5          REAL      ARRAY    / /            37   E6      REAL     ARRAY    / /
F           REAL                               0   K       INTEGER           F.
PAR         REAL      *UNUSED     F.P.         0   X       REAL     ARRAY    F.

BLOCKS    LENGTH
/ /           37

ICS
AM LENGTH                 323B     211
ANK COMMON LENGTH          45B      37
  52000B CM USED
```

B3 SUBROUTINE ZSYSTM

The subroutine ZSYSTM uses Brown's method [16] which is at least quadratically convergent and requires only $N^2/2 + 3N/2$ function evaluations per iterative step, as compared with $N^2 + N$ evaluations for Newton's method (N is the number of simultaneous equations).

It should be pointed out that ZSYSTM will terminate processing if a root is not found within ITMAX (the maximum allowable number of iterations) iterations and/or if the Jacobian matrix of the system of equations becomes computationally singular. In this case, a different initial approximation should be tried and/or the equations should be studied to see if some of the equations or variables can be eliminated or solved for, in terms of others.

B4 COMPUTER PROGRAM - METHOD 2

The computations have been checked using the MAP 6000 for matrix inversion and matrix multiplication. The roots of the three polynomials of equations (5.39), (5.41) and (5.43) have been calculated using the subroutine ZPOLR.

It is worth mentioning that equations (5.39), (5.41) and (5.43) are cubics and therefore their solutions give three values for each of the deflection coefficients, δ_1, δ_2 and δ_3. Some of these values correspond to stable equilibrium, while the remaining values are either imaginary or correspond to unstable equilibrium. These values appear when using the subroutine ZPOLR, and have been omitted.

B5 COMPUTATION

All the solutions presented were obtained with a specified degree of accuracy in the iterative solution of the three cubic equations. The specified accuracy was EPS = .000 000 01, where EPS $\leq$ .000 000 1 indicates that the root X(1), X(2) and X(3) is accepted (only if the maximum absolute value of $f_i(X) = 0$ is less than or equal to EPS).

For the matrix inversion routine, the accuracy IDGT was chosen equal to 7. This guarantees that the output resulting from the matrix inversion is accurate within - at least - 7 digits which were unchanged after improvement.

The maximum intensities of pressure for which computations were carried out in the present investigations are, in most cases, much higher than those in the previous solutions of large deflections of unstiffened plates. The maximum value of the non-dimensional pressure intensity (qa^4/Et^4) in the present solutions is 400. This value being chosen such that the ratio of the central deflection to the plate thickness (W/t) is at least 3.3 at the maximum pressure. For a single solution, the number of iterations required was between 3 and 7.

The CDC computer, employed to obtain the solutions, required about 4.6 seconds compilation time and for a typical case presented here, the time required for computation and output of the results, for all the ten pressure levels, was about 4.6 seconds.